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Abstract

This dissertation explores several classes of nonlinear difference equations and nonlinear fuzzy difference equations, with a primary emphasis on the qualitative properties of their solutions.

The work begins with a comprehensive study of nonlinear higher-order fuzzy difference equations. Under suitable initial conditions, we prove the existence and uniqueness of positive fuzzy solutions. In addition, we establish conditions ensuring their boundedness, persistence, and convergence to a single equilibrium point. The convergence rate is analyzed, and numerical simulations are provided to support and validate the theoretical results.

The concluding part addresses a three-dimensional nonlinear system of difference equations involving arbitrary constants and non-identical parameters. By performing a linearization around its equilibrium, we investigate the local dynamics and derive sufficient criteria for asymptotic stability. Moreover, it is shown that the solutions may exhibit oscillatory, bounded, or persistent behaviors depending on the parameter choices and initial conditions.

Keywords: Fuzzy difference equations, fuzzy numbers, systems of difference equations, equilibrium points, local stability, asymptotic behavior, rate of convergence.

Résumé

Cette thèse examine plusieurs classes d'équations aux différences non linéaires et d'équations aux différences floues non linéaires, en mettant l'accent sur les propriétés qualitatives de leurs solutions.

Le travail débute par une étude approfondie des équations aux différences floues non linéaires d'ordre supérieur. Sous des conditions initiales appropriées, nous établissons l'existence et l'unicité de solutions floues positives. Nous déterminons également les conditions garantissant leur bornitude, leur persistance et leur convergence vers un point d'équilibre unique. Le rythme de convergence est analysé, et des simulations numériques sont proposées afin d'appuyer et de valider les résultats théoriques.

La dernière partie est consacrée à un système non linéaire tridimensionnel d'équations aux différences comportant des constantes arbitraires et des paramètres distincts. En linéarisant le système autour de son équilibre, nous étudions la dynamique locale et formulons des critères suffisants de stabilité asymptotique. De plus, il est montré que les solutions peuvent présenter des comportements oscillatoires, bornés ou persistants, en fonction du choix des paramètres et des conditions initiales.

Mots-clés: Equations aux différences floues, nombres flous, systèmes d'équations aux différences, points d'équilibre, stabilité locale, comportement asymptotique, taux de convergence.

ملخــص

تُكرَّس هذه الأطروحة لدراسة عدد من الفئات من معادلات الفروق غير الخطية ومعادلات الفروق الضبابية غير الخطية، مع التركيز على التحليل النوعي لسلوك حلولها.

يُعنى الجزء الأول من هذا العمل بالتحليل الديناميكي لمعادلات الفروق الضبابية من الرتب العليا، حيث يتم إثبات وجود ووحدانية الحلول الضبابية الموجبة تحت شروط ابتدائية ملائمة. كما يتم برهان محدودية هذه الحلول واستمراريتها وتقاربها نحو نقطة توازن وحيدة. علاوةً على ذلك، يُبحث في معدل التقارب، وتُقدَّم أمثلة عددية لتوضيح النتائج النظرية ودعمها بالتطبيقات الحسابية.

أما في الجزء الأخير من الأطروحة، فيُخصَّص لدراسة نظام غير خطي ثلاثي الأبعاد من معادلات الفروق يتميّز بمعاملات كيفية ووسائط غير متماثلة. ومن خلال تحليل خطي للنظام حول نقطة التوازن، تُدرس الديناميكيات المحلية وتُستخلص شروط كافية للاستقرار المحلي. كما يُبيَّن أن الحلول تُظهر سلوكًا تذبذبيًا محدودًا ومستمرًا، يتأثر بقيم الوسائط وبطبيعة الشروط الابتدائية.

الكلمات الأساسية: معادلات الفروق الضبابية، الأعداد الضبابية، أنظمة المعادلات الفرقية، نقاط التوازن، الاستقرار المحلي، السلوك التقاربي، معدل التقارب

Table of contents

List of Abbreviations

Fuz. Diff. Eq. fuzzy difference equation

Fuz. Diff. Eqs. fuzzy difference equations

Fuz. Num. fuzzy number

Fuz. Nums. fuzzy numbers

Pos. Fuz. Num. positive fuzzy number

Pos. Fuz. Nums. positive fuzzy numbers

Fuz. Sol. fuzzy solution

Fuz. Sols. fuzzy solutions

Pos. Fuz. Sol. positive fuzzy solution

Pos. Fuz. Sols. positive fuzzy solutions

General introduction

The concept of uncertainty plays a central role in modern science, particularly in applied mathematics. Traditionally, uncertainty has been addressed through probability theory and statistics, which provide tools to model random phenomena governed by well-defined probability laws. However, in many contexts, uncertainty does not arise solely from randomness but also from vagueness or incompleteness of information. This type of uncertainty is especially difficult to capture using classical probabilistic approaches.

In this context, the theory of fuzzy sets, first proposed by Zadeh in 1965, provides a framework to represent imprecise or linguistic data by degrees of membership ranging between 0 and 1. Unlike probabilistic methods, which measure the frequency of occurrence of an event, fuzzy logic evaluates the degree of truth of a statement. This approach has opened new perspectives in the modeling of uncertain and complex systems.

Fuzzy difference equations are a natural extension of classical difference equations, which are widely used to model discrete dynamical systems. They allow uncertainty related to initial conditions, parameters, or even the structure of the system to be incorporated, thus providing a more flexible and realistic framework for describing the temporal evolution of many phenomena. A number of these contributions are documented in earlier studies (see, for instance, [4], [8], [10], [11], [12], [75], [76], [77], [79]).

The usefulness of fuzzy difference equations has been demonstrated in various fields:

In economics and finance, they model the evolution of economic variables (prices, interest rates, production) when available data are imprecise or subject to qualitative judgments.

In biology and ecology, they serve as tools to describe population dynamics when biological parameters (reproduction rates, mortality rates, available resources) are not precisely known.

In engineering and information sciences, they are applied to the modeling and control of systems where sensors deliver noisy or uncertain data.

In contrast, growing attention has been directed toward the theory of difference equations by researchers across diverse fields. Consequently, an extensive literature has emerged that investigates both difference equations and the systems derived from them. Several of these works can be found in previous studies (see, for instance, [[1], [2], [6], [22], [33], [39], [61], [62]]).

The main contribution of this thesis lies in a thorough investigation of the qualitative aspects of nonlinear higher-order Fuz. Diff. Eqs., as well as systems involving higher-order difference equations. Specifically, we present results concerning the existence, uniqueness, positivity, boundedness, stability, persistence, convergence, and oscillatory patterns of their solutions. By combining rigorous theoretical developments with complementary numerical experiments, this work not only advances the mathematical insight into Fuz. Diff. Eqs. but also highlights their significance in scientific and engineering contexts where uncertainty naturally arises.

The first chapter is devoted to presenting the essential background material needed throughout the thesis. It provides key definitions, theorems, and lemmas that serve as the foundation for the study of Fuz. Diff. Eqs., classical difference equations, and systems of such equations.

The second chapter, inspired by the work of [77], focuses on examining the existence, positivity, and uniqueness of fuzzy Solution to a class of fuzzy difference equations

$$y_{n+1} = \beta + \frac{\gamma y_n}{y_{n-\kappa}^2}, \quad n = 0, 1, \cdots$$

where $y_{-\kappa}, y_{-\kappa+1}, \cdots, y_0$, with $\kappa \in \{2, 3, \cdots\}$ and β, γ are Pos. Fuz. Nums.

Furthermore, we investigate the boundedness and convergence of Pos. Fuz. Sols., along with an examination of their convergence speed. To validate these theoretical results, several illustrative numerical examples are provided.

In addition, we propose the concept of g-division for Fuz. Nums., which proves to be essential in establishing new results concerning the existence, uniqueness, and stability of Pos. Fuz. Sols. in the framework of higher-order Fuz. Diff. Eqs..

Similar to Chapter 2, the third chapter is dedicated to analyzing the qualitative properties of a discrete nonlinear higher-order Fuz. Diff. Eq..

$$\omega_{n+1} = \frac{A\omega_{n-k}}{1 + \omega_{n-1}^{p}}, \quad n \in \mathbb{N}_0,\ k \in \mathbb{N}_2,$$

In this context, A, along with the initial data $\omega_{-k}, \omega_{-k+1}, \ldots, \omega_0$, is regarded as belonging to the class of Pos. Fuz. Nums.. We derive results concerning the existence, positivity, and uniqueness of solutions, and we further investigate the boundedness and persistence of positive solutions by presenting sufficient conditions that ensure their stability and convergence. These analytical results are then supported by several numerical examples.

The final chapter is devoted to a comprehensive and in-depth investigation of the **local asymptotic stability**, **boundedness**, **persistence**, and **oscillatory behavior** of positive solutions associated with a nonlinear higher-order three-dimensional system of rational difference equations. The system under consideration is expressed as follows:

$$\begin{cases} \eta_{n+1} = \varphi + \dfrac{\eta_{n-m}^{q}}{\omega_n^{q}}, \\ \omega_{n+1} = \varphi + \dfrac{\omega_{n-m}^{q}}{\xi_n^{q}}, \\ \xi_{n+1} = \varphi + \dfrac{\xi_{n-m}^{q}}{\eta_n^{q}}, \end{cases}$$

where $m \geq 2$, $\eta_i, \omega_i, \xi_i \in (0, +\infty)$ for $i \in \{-m, -m+1, \ldots, 0\}$, $\varphi > 0$, and $q \geq 1$.

This class of nonlinear systems provides a versatile mathematical framework for exploring interconnected discrete dynamical processes characterized by mutual dependence among variables and by the influence of time-delayed terms. Such models can effectively describe real phenomena in *biological*, *economic*, and *engineering* contexts, where delayed interactions or nonlinear feedback mechanisms play a fundamental role.

In this chapter, we begin by determining the equilibrium points of the system and examining their existence and uniqueness under appropriate assumptions on the parameters. Subsequently, by applying linearization techniques around each equilibrium point, we derive the Jacobian matrix and analyze its eigenvalues, thereby establishing rigorous criteria for local asymptotic stability. These analytical results allow us to characterize the conditions under which small perturbations in the initial states or parameters lead either to convergence toward equilibrium, to persistent bounded oscillations, or to instability.

Furthermore, we investigate the rate of convergence toward equilibrium in stable regimes, as well as the persistence properties ensuring that all solution components remain strictly positive and bounded away from zero. The possibility of oscillatory and periodic behaviors is also explored, emphasizing how delicate changes in system parameters can significantly alter its long-term dynamics.

Finally, a series of numerical simulations are presented to validate the theoretical findings and to illustrate the diverse dynamical patterns exhibited by the system, including monotonic convergence, quasi-periodic oscillations, and sustained persistence. The overall analysis thus provides not only a detailed qualitative description of the system's behavior but also a methodological foundation that can be extended to broader classes of nonlinear discrete models involving multiple interrelated variables.

This concluding study highlights the interplay between theoretical rigor and numerical experimentation, offering valuable insights into the stability mechanisms and bifurcation phenomena that arise in nonlinear autonomous systems of difference equations.

In summary, this thesis offers a unified and rigorous framework for analyzing the qualitative behavior of nonlinear fuzzy and crisp difference equations. By establishing conditions for the existence, uniqueness, boundedness, and stability of positive fuzzy solutions, it bridges the gap between theoretical advances in fuzzy mathematics and their practical applications in modeling uncertain dynamical processes. The results obtained herein not only contribute to the mathematical theory of discrete systems but also provide analytical tools and insights applicable to various real-world problems characterized by imprecision and incomplete data.

Chapter 1

Preliminaries and definitions

1.1 Nonlinear difference equations

Nonlinear difference equations constitute an important class of mathematical models that are widely employed in various scientific and engineering disciplines to describe complex and diverse phenomena. Unlike their linear counterparts, these equations involve nonlinear terms, which often lead to richer dynamics but also introduce significant challenges in both analysis and computation. Because of their flexibility, nonlinear difference equations are frequently used to capture behaviors such as oscillations, bifurcations, and stability changes that cannot be adequately explained by linear models. In what follows, this section is devoted to a detailed study of such equations, with particular emphasis on their qualitative properties.

Let $\mathcal{I} \subseteq \mathbb{R}$ and $f : \mathcal{I}^{k+1} \to \mathcal{I}$ be a well defined function.

Definition 1.1.1 *A difference equation of order* $(k+1)$*,*

$$\zeta_{n+1} = f(\zeta_n, \zeta_{n-1}, \ldots, \zeta_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

with $\zeta_0, \zeta_{-1}, \ldots, \zeta_{-k} \in \mathcal{I}$, is said to be nonlinear if it is not of the form

$$\zeta_{n+k+1} + s_1\zeta_{n+k} + \cdots + s_k\zeta_n = g(n), \; n \in \mathbb{N} \tag{1.2}$$

Where $s_1, s_2 \ldots s_k$ are real or complex constants

Definition 1.1.2 *A point $\bar{\zeta} \in \mathcal{I}$ is called an* **equilibrium point** *of equation* (1.1) *if*

$$\bar{\zeta} = f(\bar{\zeta}, \bar{\zeta}, \ldots, \bar{\zeta}).$$

Equivalently,

$$\zeta_n = \bar{\zeta}, \quad \forall n \geq -k.$$

Definition 1.1.3 *An interval $\mathcal{J} \subseteq \mathcal{I}$ is said to be* **an invariant interval** *of equation* (1.1) *if*

$$\zeta_{-k}, \zeta_{-k+1}, \cdots, \zeta_0 \in \mathcal{J} \Rightarrow \zeta_n \in \mathcal{J}, \;\; \forall n > n_0.$$

1.1.1 The analysis of equilibrium point stability in nonlinear difference equations

Stability is one of the most critical properties to investigate in the study of dynamical systems, as it provides essential insights into the long-term behavior of solutions. Understanding whether small perturbations in the initial conditions lead to bounded or divergent trajectories is fundamental for predicting system performance and reliability. In many cases, obtaining an exact analytical solution is either extremely difficult or entirely impossible due to the system's complexity or nonlinearity. Therefore, researchers often rely on qualitative methods of analysis to examine the stability characteristics, such as linearization around equilibrium points, spectral analysis, or Lyapunov's direct method. These approaches allow us to infer the general behavior of the system without requiring an explicit closed-form solution.

Definition 1.1.4 *Suppose that $\bar{\zeta}$ is an equilibrium point of* (1.1),

1. *$\bar{\zeta}$ is considered* **locally stable** *if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall\ \zeta_{-k}, \zeta_{-k+1}, \ldots, \zeta_0 \in \mathcal{I}:\ \ |\zeta_{-k} - \bar{\zeta}| + |\zeta_{-k+1} - \bar{\zeta}| + \cdots + |\zeta_0 - \bar{\zeta}| < \delta,$$

then

$$|\zeta_n - \bar{\zeta}| < \varepsilon,\ \ \forall n \geq -k.$$

2. *$\bar{\zeta}$ is said to be* **locally asymptotically stable** *if*

- *$\bar{\zeta}$ is locally stable.*
- $\exists \gamma > 0,\ \forall\ \zeta_{-k}, \zeta_{-k+1}, \ldots, \zeta_0 \in \mathcal{I}:\ \ |\zeta_{-k} - \bar{\zeta}| + |\zeta_{-k+1} - \bar{\zeta}| + \cdots + |\zeta_0 - \bar{\zeta}| < \gamma$, *so*

$$\lim_{n\to+\infty} \zeta_n = \bar{\zeta}.$$

3. *$\bar{\zeta}$ is considered* **globally attractive** *if*

$$\forall\ \zeta_{-k}, \zeta_{-k+1}, \ldots, \zeta_0 \in \mathcal{I},\ \ \lim_{n\to+\infty} \zeta_n = \bar{\zeta}.$$

4. *$\bar{\zeta}$ is considered* **globally asymptotically stable** *if*

- *$\bar{\zeta}$ is locally stable.*
- *$\bar{\zeta}$ is globally attractive.*

5. *$\bar{\zeta}$ is considered* **unstable** *if it does not exhibit local stability.*

Definition 1.1.5 *The* ***linear difference equation associated with*** (1.1) *is defined as follows:*

$$y_{n+1} = s_0 y_n + s_1 y_{n-1} + \cdots + s_k y_{n-k}, \tag{1.3}$$

where

$$s_i = \frac{\partial f}{\partial u_i}(\bar{\zeta}, \bar{\zeta}, \ldots, \bar{\zeta}), \qquad \text{for } i = \overline{0, k},$$

and

$$\begin{array}{rccc} f: & \mathcal{I}^{k+1} & \longrightarrow & \mathcal{I} \\ & (u_0, u_1, \ldots, u_k) & \longmapsto & f(u_0, u_1, \ldots, u_k). \end{array}$$

Before introducing the main stability criteria, it is important to recall some classical results that form the cornerstone of the qualitative analysis of nonlinear difference equations. In particular, when direct analytical solutions are difficult or impossible to obtain, the study of stability through linearization provides a powerful tool for understanding the local behavior of solutions near an equilibrium point. The idea is to approximate the nonlinear equation by its linear counterpart in a small neighborhood of equilibrium and to analyze the characteristic roots of the corresponding linearized system. The following theorems summarize the essential results that will be used throughout this work to determine the local asymptotic stability or instability of equilibrium points.

Theorem 1.1.1 [53] **(Stability via Linearization)**

1. *If every root of the characteristic polynomial corresponding to the associated linear difference equation lies inside the open unit disk* $|\lambda| < 1$*, then the equilibrium point of* (1.1) *is locally asymptotically stable.*

2. *If at least one root of this characteristic polynomial satisfies* $|\lambda| > 1$*, then the equilibrium point of* (1.1) *is unstable.*

While the previous results establish stability conditions based on the linear approximation of nonlinear difference equations, it is also valuable to have general criteria that guarantee global convergence without explicitly relying on linearization. Such results are particularly useful when the nonlinearities involved prevent the use of spectral analysis or when monotonicity properties can be exploited to ensure convergence. The next theorem provides sufficient conditions for the existence and uniqueness of an equilibrium point and demonstrates that every solution of the nonlinear difference equation converges to this equilibrium under appropriate monotonicity assumptions on the mapping function.

Theorem 1.1.2 [53] *Let $f : [u,v]^{k+1} \to [u,v]$ be a continuous mapping, where k is a positive integer and $[u,v]$ is an interval of real numbers. Consider the difference equation*

$$\zeta_{n+1} = f\Big(\zeta_n, \zeta_{n-1}, \ldots, \zeta_{n-k}\Big), \quad n = 0, 1, \ldots. \tag{1.4}$$

Assume that f satisfies the following conditions:

i. *For every integer i with $1 \leq i \leq k+1$, the function $f(\zeta_1, \zeta_2, \ldots, \zeta_{k+1})$ is weakly monotone in ζ_i when the remaining variables are fixed.*

ii. *If (m, M) solves the system*

$$m = f(m_1, m_2, \ldots, m_{k+1}) \quad \textit{and} \quad M = f(M_1, M_2, \ldots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \ldots, k+1$,

$$m_i = \begin{cases} m, & \textit{if } f \textit{ is nondecreasing in } \zeta_i, \\ M, & \textit{if } f \textit{ is nonincreasing in } \zeta_i, \end{cases}$$

and

$$M_i = \begin{cases} M, & \textit{if } f \textit{ is nondecreasing in } \zeta_i, \\ m, & \textit{if } f \textit{ is nonincreasing in } \zeta_i. \end{cases}$$

Then, the difference equation (1.4) admits a unique equilibrium point $\bar{\zeta}$, and every solution of (1.4) converges to $\bar{\zeta}$.

Now consider the scalar k-th order linear difference equation

$$\zeta_{n+k} + s_1(n)\zeta_{n+k-1} + \cdots + s_k(n)\zeta_n = 0, \tag{1.5}$$

where k is a positive integer and $s_i : \mathbb{Z}^+ \to \mathbb{C}$ for $i = 1, \ldots, k$. Suppose that the limits

$$q_i = \lim_{n\to\infty} s_i(n), \quad i = 1, \ldots, k,$$

exist in $\mathbb{C}$.

Then, the limiting equation associated with (1.5) *is*

$$\zeta_{n+k} + q_1\zeta_{n+k-1} + \cdots + q_k\zeta_n = 0. \tag{1.6}$$

1.2 Systems of nonlinear difference equations

In the study of systems of nonlinear difference equations, stability analysis plays a fundamental role in understanding the evolution and behavior of discrete-time dynamical models. Such systems often arise in various scientific and engineering fields, including biology, economics, and control theory, where processes evolve in discrete steps rather than continuously. Due to the nonlinear nature of these equations, obtaining exact analytical solutions is generally challenging or even impossible. Consequently, researchers frequently turn to qualitative methods to investigate stability properties. Techniques such as linearization around equilibrium points, the use of characteristic equations, and spectral radius conditions are commonly employed to determine whether small deviations from equilibrium grow or decay over time. Through these qualitative approaches, one can gain valuable insights into the long-term dynamics of nonlinear discrete systems without the need for explicit solutions.

Suppose $f^{(1)}, f^{(2)}, \ldots, f^{(p)}$ denote functions that are continuously differentiable, such that

$$f^{(i)} : \mathcal{I}_1^{k+1} \times \mathcal{I}_2^{k+1} \times \cdots \times \mathcal{I}_p^{k+1} \to \mathcal{I}_i^{k+1}, \;\; i = \overline{1,p},$$

with $\mathcal{I}_i,\ i = \overline{1,p}$ present real intervals.

Consider the following p-dimensional system

$$\begin{cases} \zeta_{n+1}^{(1)} &= f^{(1)}\left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots\ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right) \\ \zeta_{n+1}^{(2)} &= f^{(2)}\left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots\ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right) \\ \quad\vdots & \\ \zeta_{n+1}^{(p)} &= f^{(p)}\left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots\ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right) \end{cases} \tag{1.7}$$

with $n,\ k \in \mathbb{N}_0,\ \left(\zeta_{-k}^{(i)}, \zeta_{-k+1}^{(i)}, \ldots, \zeta_0^{(i)}\right) \in \mathcal{I}_i^{k+1},\ \ i = \overline{1,p}.$

Let us define the function

$$F : \mathcal{I}_1^{(k+1)} \times \mathcal{I}_2^{(k+1)} \times \cdots \times \mathcal{I}_p^{(k+1)} \longrightarrow \mathcal{I}_1^{(k+1)} \times \mathcal{I}_2^{(k+1)} \times \cdots \times \mathcal{I}_p^{(k+1)}$$

as follows

$$F(\mathcal{Z}) = \left(f_0^{(1)}(\mathcal{Z}), f_1^{(1)}(\mathcal{Z}), \ldots, f_k^{(1)}(\mathcal{Z}), f_0^{(2)}(\mathcal{Z}), f_1^{(2)}(\mathcal{Z}), \ldots, f_k^{(2)}(\mathcal{Z}), \ldots, f_0^{(p)}(\mathcal{Z}), f_1^{(p)}(\mathcal{Z}), \ldots, f_k^{(p)}(\mathcal{Z})\right),$$

with

$$\mathcal{Z} = \left(u_0^{(1)}, u_1^{(1)}, \ldots, u_k^{(1)}, u_0^{(2)}, u_1^{(2)}, \ldots, u_k^{(2)}, \ldots, u_0^{(p)}, u_1^{(p)}, \ldots, u_k^{(p)}\right)^T,$$

$$f_0^{(i)}(\mathcal{Z}) = f^{(i)}(\mathcal{Z}), \quad f_1^{(i)}(\mathcal{Z}) = u_0^{(i)}, \ldots, f_k^{(i)}(\mathcal{Z}) = u_{k-1}^{(i)},\ \ i = \overline{1,p}.$$

Let us set

$$\mathcal{Z}_n = \left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right)^T.$$

Thus, system (1.7) can be expressed as the following one

$$\mathcal{Z}_{n+1} = F(\mathcal{Z}_n),\ \ n = 0, 1, 2, \ldots \tag{1.8}$$

that is to say

$$\begin{cases} \zeta_{n+1}^{(1)} &= f^{(1)}\left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots\ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right) \\ \zeta_n^{(1)} &= \zeta_n^{(1)} \\ &\vdots \\ \zeta_{n-k+1}^{(1)} &= \zeta_{n-k+1}^{(1)} \\ \zeta_{n+1}^{(2)} &= f^{(2)}\left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots\ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right) \\ \zeta_n^{(2)} &= \zeta_n^{(2)} \\ &\vdots \\ \zeta_{n-k+1}^{(2)} &= \zeta_{n-k+1}^{(2)} \\ &\vdots \\ \zeta_{n+1}^{(p)} &= f^{(p)}\left(\zeta_n^{(1)}, \zeta_{n-1}^{(1)}, \ldots, \zeta_{n-k}^{(1)}, \zeta_n^{(2)}, \zeta_{n-1}^{(2)}, \ldots, \zeta_{n-k}^{(2)}, \ldots\ldots, \zeta_n^{(p)}, \zeta_{n-1}^{(p)}, \ldots, \zeta_{n-k}^{(p)}\right) \\ \zeta_n^{(p)} &= \zeta_n^{(p)} \\ &\vdots \\ \zeta_{n-k+1}^{(p)} &= \zeta_{n-k+1}^{(p)} \end{cases}$$

Definition 1.2.1

1. $(\overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}})$ *is considered* **an equilibrium point** *of system* (1.7) *if*

$$\overline{\zeta^{(1)}} = f^{(1)}\left(\overline{\zeta^{(1)}}, \overline{\zeta^{(1)}}, \ldots, \overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}}, \overline{\zeta^{(p)}}, \ldots, \overline{\zeta^{(p)}}\right),$$

$$\overline{\zeta^{(2)}} = f^{(2)}\left(\overline{\zeta^{(1)}}, \overline{\zeta^{(1)}}, \ldots, \overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}}, \overline{\zeta^{(p)}}, \ldots, \overline{\zeta^{(p)}}\right),$$

$$\vdots \qquad\qquad \vdots$$

$$\overline{\zeta^{(p)}} = f^{(p)}\left(\overline{\zeta^{(1)}}, \overline{\zeta^{(1)}}, \ldots, \overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}}, \overline{\zeta^{(p)}}, \ldots, \overline{\zeta^{(p)}}\right).$$

2. $\overline{\mathcal{Z}} = \left(\overline{\zeta^{(1)}}, \overline{\zeta^{(1)}}, \ldots, \overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}}, \overline{\zeta^{(p)}}, \ldots, \overline{\zeta^{(p)}}\right) \in \mathcal{I}_1^{k+1} \times \mathcal{I}_2^{k+1} \times \ldots \times \mathcal{I}_p^{k+1}$ *represents an equilibrium of system* (1.8) *if*

$$\overline{\mathcal{Z}} = F(\overline{\mathcal{Z}}).$$

1.2.1 The stability of equilibrium points of nonlinear difference equations system

The study of stability in difference equations plays a fundamental role in understanding their behavior, as it indicates how solutions evolve when initial conditions or parameters are perturbed. Stability analysis makes it possible to determine whether a system will remain near equilibrium, display oscillations, or become unstable as time advances. Such analysis is crucial for evaluating the reliability and predictability of the system. By investigating stability, one gains valuable insights into the long-term dynamics of the system and its potential responses to future variations.

Definition 1.2.2 *Let $\overline{\mathcal{Z}}$ be an equilibrium point of system* (1.8)*, and let $\|\cdot\|$ denote a norm (for instance, the Euclidean norm).*

1. *The equilibrium $\overline{\mathcal{Z}}$ is said to be* stable *(or* locally stable*) if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $\|\mathcal{Z}_0 - \overline{\mathcal{Z}}\| < \delta$, it follows that $\|\mathcal{Z}_n - \overline{\mathcal{Z}}\| < \varepsilon$ for all $n \geq 0$.*

2. *The point $\overline{\mathcal{Z}}$ is called* asymptotically stable *(or* locally asymptotically stable*) if it is stable and, in addition, there exists $\gamma > 0$ such that whenever $\|\mathcal{Z}_0 - \overline{\mathcal{Z}}\| < \gamma$, one has*

$$\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}}, \quad n \to +\infty.$$

3. *The equilibrium $\overline{\mathcal{Z}}$ is said to be* globally attractive *(or globally attractive with respect to a domain of attraction $G \subseteq I_1^{k+1} \times I_2^{k+1} \times \cdots \times I_p^{k+1}$) if, for any initial state $\mathcal{Z}_0$ (respectively, for every $\mathcal{Z}_0 \in G$), we have*

$$\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}}, \quad n \to +\infty.$$

4. *The equilibrium $\overline{\mathcal{Z}}$ is* globally asymptotically stable *(or globally asymptotically stable with respect to G) if it is locally stable and, for every initial condition $\mathcal{Z}_0$ (respectively,*

for all $\mathcal{Z}_0 \in G$), it holds that

$$\mathcal{Z}_n \longrightarrow \overline{\mathcal{Z}}, \quad n \to +\infty.$$

5. *Finally, $\overline{\mathcal{Z}}$ is said to be* unstable *whenever the condition of local stability is not satisfied.*

Remark 1.2.1 *It is straightforward to observe that $\left(\overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}}\right) \in \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_p$ is an equilibrium of* (1.7) *if and only if*

$$\overline{\mathcal{Z}} = \left(\overline{\zeta^{(1)}}, \overline{\zeta^{(1)}}, \ldots, \overline{\zeta^{(1)}},\ \overline{\zeta^{(2)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(2)}},\ \ldots,\ \overline{\zeta^{(p)}}, \overline{\zeta^{(p)}}, \ldots, \overline{\zeta^{(p)}}\right) \in \mathcal{I}_1^{k+1} \times \mathcal{I}_2^{k+1} \times \cdots \times \mathcal{I}_p^{k+1}$$

is an equilibrium of (1.8).

Definition 1.2.3 (Associated linear system) *We call linear system associated with system* (1.8) *around $\overline{\mathcal{Z}} = \left(\overline{\zeta^{(1)}}, \overline{\zeta^{(1)}}, \ldots, \overline{\zeta^{(1)}}, \overline{\zeta^{(2)}}, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(2)}}, \ldots, \overline{\zeta^{(p)}}, \overline{\zeta^{(p)}}, \ldots, \overline{\zeta^{(p)}}\right)$,*

the system

$$\mathcal{Z}_{n+1} = J_F \mathcal{Z}_n, \;\; n = 0, 1, 2, \ldots$$

where J_F represents the Jacobian matrix of F evaluated at the equilibrium point $\overline{\mathcal{Z}}$, and is given by

$$
J_F = \begin{pmatrix}
\frac{\partial f_0^{(1)}}{\partial u_0^{(1)}} & \frac{\partial f_0^{(1)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_k^{(1)}} & \frac{\partial f_0^{(1)}}{\partial u_0^{(2)}} & \frac{\partial f_0^{(1)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_0^{(p)}} & \frac{\partial f_0^{(1)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_0^{(1)}}{\partial u_k^{(p)}} \\
\frac{\partial f_1^{(1)}}{\partial u_0^{(1)}} & \frac{\partial f_1^{(1)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_k^{(1)}} & \frac{\partial f_1^{(1)}}{\partial u_0^{(2)}} & \frac{\partial f_1^{(1)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_0^{(p)}} & \frac{\partial f_1^{(1)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_1^{(1)}}{\partial u_k^{(p)}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k^{(1)}}{\partial u_0^{(1)}} & \frac{\partial f_k^{(1)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_k^{(1)}} & \frac{\partial f_k^{(1)}}{\partial u_0^{(2)}} & \frac{\partial f_k^{(1)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_0^{(p)}} & \frac{\partial f_k^{(1)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_k^{(1)}}{\partial u_k^{(p)}} \\
\frac{\partial f_0^{(2)}}{\partial u_0^{(1)}} & \frac{\partial f_0^{(2)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_k^{(1)}} & \frac{\partial f_0^{(2)}}{\partial u_0^{(2)}} & \frac{\partial f_0^{(2)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_0^{(p)}} & \frac{\partial f_0^{(2)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_0^{(2)}}{\partial u_k^{(p)}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_0^{(p)}}{\partial u_0^{(1)}} & \frac{\partial f_0^{(p)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_k^{(1)}} & \frac{\partial f_0^{(p)}}{\partial u_0^{(2)}} & \frac{\partial f_0^{(p)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_0^{(p)}} & \frac{\partial f_0^{(p)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_0^{(p)}}{\partial u_k^{(p)}} \\
\frac{\partial f_1^{(p)}}{\partial u_0^{(1)}} & \frac{\partial f_1^{(p)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_k^{(1)}} & \frac{\partial f_1^{(p)}}{\partial u_0^{(2)}} & \frac{\partial f_1^{(p)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_0^{(p)}} & \frac{\partial f_1^{(p)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_1^{(p)}}{\partial u_k^{(p)}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k^{(p)}}{\partial u_0^{(1)}} & \frac{\partial f_k^{(p)}}{\partial u_1^{(1)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_k^{(1)}} & \frac{\partial f_k^{(p)}}{\partial u_0^{(2)}} & \frac{\partial f_k^{(p)}}{\partial u_1^{(2)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_k^{(2)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_0^{(p)}} & \frac{\partial f_k^{(p)}}{\partial u_1^{(p)}} & \cdots & \frac{\partial f_k^{(p)}}{\partial u_k^{(p)}}
\end{pmatrix},
$$

such that

$$
f_j^{(i)} = f_j^{(i)}\left(\overline{\mathcal{Z}}\right), \; i = \overline{1,p}, \; j = \overline{0,k}.
$$

In the case of multidimensional systems of nonlinear difference equations, stability analysis becomes more intricate due to the interaction between multiple state variables. Nevertheless, the fundamental idea remains similar to the scalar case: the local behavior of the system near an equilibrium point can be characterized by examining the eigenvalues of the Jacobian matrix evaluated at that equilibrium. This linearization approach provides a convenient and rigorous method for determining whether small deviations from equilibrium decay or amplify over time. The following theorem presents the main criterion for assessing local asymptotic stability and instability of equilibrium points in such systems.

Theorem 1.2.1 [53] **(Stability through Linearization)**

1. *If all eigenvalues of J_F lie strictly inside the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{\mathcal{Z}}$ is locally asymptotically stable.*

2. *If at least one eigenvalue of J_F has modulus greater than one, then the equilibrium point $\overline{\mathcal{Z}}$ is unstable.*

1.2.2 Rate of convergence

The rate of convergence describes how quickly the solutions of a difference equation approach their equilibrium points. Beyond determining stability, this measure provides insight into the system's dynamic responsiveness. The following classical results, due to Perron, are useful for estimating such rates in linear and asymptotically linear systems.

Here, we are going to give two important propositions which assists in estimating the rate of convergence.

Let's consider the following difference equations system

$$\mathcal{Z}_{n+1} = (\mathcal{A} + \mathcal{B}(n))\,\mathcal{Z}_n, \tag{1.9}$$

with $\mathcal{Z}_n$ represents a vector of dimension m, $\mathcal{A} \in C^{m\times m}$ represents a constant matrix, and $\mathcal{B} : \mathbb{Z}^+ \to C^{m\times m}$ represents a matrix function that satisfies the condition

$$\|\mathcal{B}(n)\| \to 0 \tag{1.10}$$

as n approaches infinity, where $\|.\|$ signifies any matrix norm corresponding to the vector norm

$$\|(\zeta_1, \zeta_2, \ldots, \zeta_m)\| = \sqrt{\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_m^2}.$$

Proposition 1.2.1 [63] **(The 1^{st} Perron's Theorem)**

Suppose condition (1.10) is met. If $\mathcal{Z}_n$ represents a solution to system (1.9), then either

$\mathcal{Z}_n = 0$ *for all sufficiently large n or*

$$\rho = \lim_{n\to\infty} (\|\mathcal{Z}_n\|)^{\frac{1}{n}} \tag{1.11}$$

exists and is equal to the modulus of one of the eigenvalues of $\mathcal{A}$.

Proposition 1.2.2 [63] **(The 2^{nd} Perron's Theorem)**

Suppose condition (1.10) *is met. If* $\mathcal{Z}_n$ *represents a solution to system* (1.9), *then either* $\mathcal{Z}_n = 0$ *for all sufficiently large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathcal{Z}_{n+1}\|}{\|\mathcal{Z}_n\|} \tag{1.12}$$

exists and is equal to the modulus of one of the eigenvalues of $\mathcal{A}$.

1.3 Fuzzy Sets

The concept of a fuzzy subset, which is an extension of the notion of a classical subset, was introduced by Zadeh to avoid abrupt transitions from one class to another (for example, from the black class to the white class), and to allow elements not to belong completely to either one (for example, being gray), or to belong partially to each class (having a high degree of membership to the black class and a low degree of membership to the white class in the case of dark gray).

Fuzzy subsets are therefore used to describe vague and imprecise concepts, gradual properties, or uncertain events. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and several properties of these notions in the context of fuzzy sets are established.

1.3.1 Definition of a fuzzy set and fuzzy subset

Definition 1.3.1 *A fuzzy set is a class of objects characterized by a function that assigns to each object a degree of membership to this class, belonging to the interval* $[0, 1]$*.*
Thus, the closer the degree of membership is to 1, the stronger the membership.

Definition 1.3.2 *Let X be a classical reference set.*

A fuzzy subset A of X is defined by a membership function that associates to each element x of X a degree $f_A(x)$*, between* 0 *and* 1*, by which x belongs to A:*

$$f_A : X \to [0, 1]$$

Formally, the fuzzy subset A can be defined by the pair (X, f_A)*.*
If f_A *only takes the values* 0 *or* 1*, then the fuzzy subset A becomes a classical subset of X. Therefore, a classical subset is a particular case of a fuzzy subset.*

1.3.2 Notations

Let A be a fuzzy subset of an ordinary set X. We denote:

$$A = \frac{f_A(x_1)}{x_1} + \frac{f_A(x_2)}{x_2} + \cdots + \frac{f_A(x_n)}{x_n}, \quad \text{if } X = \{x_1, x_2, \ldots, x_n\}$$

where $\frac{f_A(x_i)}{x_i}$ means that the element x_i belongs to the fuzzy set A with degree $f_A(x_i)$. and $f_A(x_i)$ is the truth degree of the proposition: $x_i \in A$

If X is countable, we write:

$$A = \sum_{x \in X} \frac{f_A(x)}{x}$$

If X is continuous, we write:

$$A = \int_X \frac{f_A(x)}{x}$$

The family of all fuzzy subsets of X is denoted by $F(X)$.

Example 1.3.1 *Let X be the set of real numbers* $\mathbb{R}$*, and let A be a fuzzy subset containing the real numbers "close to 1".*

$$A = \int_{]-\infty,\frac{1}{2}]} \frac{0}{x} + \int_{]\frac{1}{2},1]} \frac{2x-1}{x} + \int_{]1,\frac{3}{2}]} \frac{-2x+3}{x} + \int_{]\frac{3}{2},+\infty[} \frac{0}{x}$$

as illustrated in the Figure 1.1

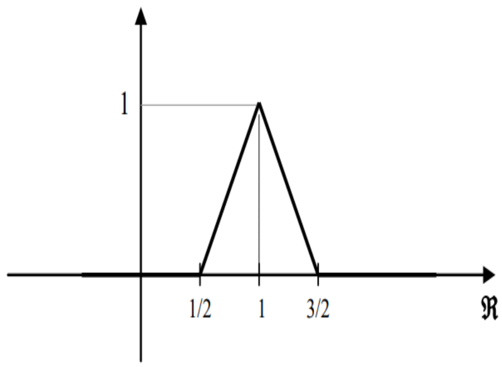


Figure 1.1: Fuzzy set close to 1

1.3.3 Operations on Fuzzy Subsets

The description of a fuzzy subset A of the reference set X corresponds to the identification of the degrees with which the property $Prop(A)$ is satisfied by the elements of X. How can we construct the fuzzy subset defined by the degrees with which $Prop(A)$ is satisfied? How can we construct the one defined by the degrees with which the two properties $Prop(A)$ and $Prop(B)$ are simultaneously satisfied? Operations on fuzzy subsets are introduced to answer such questions.

Since the concept of a fuzzy subset of X is a generalization of the notion of a classical subset, these operations are chosen in such a way that they are equivalent to classical operations when the membership functions take only the values 0 or 1.

Definition 1.3.3 *Let X be a classical reference set, and let A and B be two fuzzy subsets of X.*

1. ***Equality***

$$A = B \quad \textit{if}\ \forall x \in X,\ f_A(x) = f_B(x)$$

2. ***Inclusion***

$$A \subseteq B \quad \textit{if}\ \forall x \in X,\ f_A(x) \leq f_B(x)$$

3. ***Intersection***

$$A \cap B = C$$

such that $\forall x \in X,\ f_C(x) = \min\big(f_A(x), f_B(x)\big)$

4. ***Union***

$$A \cup B = D$$

such that $\forall x \in X,\ f_D(x) = \max\big(f_A(x), f_B(x)\big)$

5. ***Complement of a Fuzzy Subset***

$$\forall x \in X,\ f_{\bar{A}}(x) = 1 - f_A(x)$$

Properties 1.3.1 *As in classical set theory, these definitions lead to the following properties.*

- $\cup$ *and* $\cap$ *are associative.*
- $\cup$ *and* $\cap$ *are commutative.*
- $\cup$ *is distributive over* $\cap$*, and* $\cap$ *is distributive over* $\cup$*.*
- $A \cap \emptyset = \emptyset, \quad A \cap X = A$
- $A \cup \emptyset = A, \quad A \cup X = X$
- $(A^c)^c = A$
- ***De Morgan's laws:***

$$(A \cap B)^c = A^c \cup B^c, \qquad (A \cup B)^c = A^c \cap B^c$$

- *The classical properties of* ***non-contradiction***

$$A^c \cap A = \varnothing$$

and ***excluded middle***

$$A^c \cup A = X$$

are not generally satisfied. The only fuzzy sets satisfying these two properties are crisp (classical) sets.

1.3.4 Other Operators for Defining Set Operations

The operators **min**, **max**, and complement to 1 were chosen to define respectively the intersection, union, and complement of fuzzy subsets because they preserve almost all the properties of these operations in classical set theory.

However, there exist other operators that can be used when specific behaviors of operations on fuzzy subsets are desired, even if this means losing some classical properties such as non-contradiction or excluded middle.

The best-known operators are **triangular norms (t-norms)** for intersection, **triangular conorms (t-conorms)** for union, and **negations** for complement.

For example, if we define the intersection and union respectively by:

$$\forall x \in X, \quad f_{A\cap B}(x) = \max\left(f_A(x) + f_B(x) - 1, 0\right)$$

$$\forall x \in X, \quad f_{A\cup B}(x) = \min\left(f_A(x) + f_B(x), 1\right)$$

then the properties that were not satisfied with the previous choices become valid, in particular:

$$A^c \cap A = \emptyset$$

and

$$A^c \cup A = X$$

which may appear satisfactory, but this comes at the cost of losing the two idempotency properties:

$$A \cap A = A, \qquad A \cup A = A$$

1.4 Fuzzy numbers

Definition 1.4.1 [60] *A function $\mathcal{L} : \mathbb{R} \to [0,1]$ is called a Fuz. Num. if it satisfies the following properties:*

(i) $\mathcal{L}$ is normal, that is, there exists ζ such that $\mathcal{L}(\zeta) = 1$.

(ii) $\mathcal{L}$ is fuzzy convex, meaning that for all $t \in [0,1]$ and $\zeta_1, \zeta_2 \in \mathbb{R}$,

$$\mathcal{L}\Big(t\zeta_1 + (1-t)\zeta_2\Big) \geq \min\{\mathcal{L}(\zeta_1), \mathcal{L}(\zeta_2)\}.$$

(iii) $\mathcal{L}$ is upper semicontinuous.

(iv) The support of $\mathcal{L}$, defined as

$$\operatorname{supp}(\mathcal{L}) = \overline{\bigcup_{\theta \in]0,1]} [\mathcal{L}]_\theta} = \overline{\{\zeta \in \mathbb{R} : \mathcal{L}(\zeta) > 0\}},$$

is compact.

Let $\mathbb{R}_{\mathcal{F}}$ denote the set of all Fuz. Nums.. For $0 < \theta \leq 1$ and $\mathcal{L} \in \mathbb{R}_{\mathcal{F}}$:

Definition 1.4.2 [60] *The θ-cut of the Fuz. Num. $\mathcal{L}$ for $\theta \in]0,1]$ is defined by*

$$[\mathcal{L}]_\theta = \{\zeta \in \mathbb{R} : \mathcal{L}(\zeta) \geq \theta\}.$$

For $\theta = 0$, the support of $\mathcal{L}$ is given by

$$\operatorname{supp}(\mathcal{L}) = [\mathcal{L}]_0 = \overline{\{\zeta \in \mathbb{R} : \mathcal{L}(\zeta) > 0\}}.$$

Each θ-cut interval $[\mathcal{L}]_\theta$ is closed. Moreover, if $\operatorname{supp}(\mathcal{L}) \subset]0,\infty[$, then $\mathcal{L}$ is called a Pos. Fuz. Num..

Definition 1.4.3 *Let $\mathcal{L}, \mathcal{K} \in \mathbb{R}_{\mathcal{F}}$, with $[\mathcal{L}]_\theta = [\mathcal{L}_{l,\theta}, \mathcal{L}_{r,\theta}]$, $[\mathcal{K}]_\theta = [\mathcal{K}_{l,\theta}, \mathcal{K}_{r,\theta}]$, and $\lambda \in \mathbb{R}$. In the framework of standard interval arithmetic (SIA), the operations of addition, scalar multiplication, and product of Fuz. Nums. are defined as follows:*

$$[\mathcal{L} + \mathcal{K}]_\theta = [\mathcal{L}]_\theta + [\mathcal{K}]_\theta, \tag{1.13}$$

$$[\lambda \cdot \mathcal{L}]_\theta = \lambda[\mathcal{L}]_\theta, \quad \forall \theta \in [0,1], \tag{1.14}$$

$$[\mathcal{L}\mathcal{K}]_\theta = \Big[\min\{\mathcal{L}_{l,\theta}\mathcal{K}_{l,\theta}, \mathcal{L}_{l,\theta}\mathcal{K}_{r,\theta}, \mathcal{L}_{r,\theta}\mathcal{K}_{l,\theta}, \mathcal{L}_{r,\theta}\mathcal{K}_{r,\theta}\}, \max\{\mathcal{L}_{l,\theta}\mathcal{K}_{l,\theta}, \mathcal{L}_{l,\theta}\mathcal{K}_{r,\theta}, \mathcal{L}_{r,\theta}\mathcal{K}_{l,\theta}, \mathcal{L}_{r,\theta}\mathcal{K}_{r,\theta}\}\Big]. \tag{1.15}$$

In particular, if $\mathcal{L}, \mathcal{K} \in \mathbb{R}^+_{\mathcal{F}}$, then

$$[\mathcal{L}\mathcal{K}]_\theta = [\mathcal{L}_{l,\theta}\mathcal{K}_{l,\theta},\ \mathcal{L}_{r,\theta}\mathcal{K}_{r,\theta}].$$

Definition 1.4.4 *For $\mathcal{L}, \mathcal{K} \in \mathbb{R}_{\mathcal{F}}$, the metric D is defined by*

$$D(\mathcal{L}, \mathcal{K}) = \sup_{\theta\in[0,1]} \max\left\{|\mathcal{L}_{l,\theta} - \mathcal{K}_{l,\theta}|, |\mathcal{L}_{r,\theta} - \mathcal{K}_{r,\theta}|\right\}. \tag{1.16}$$

It follows that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Definition 1.4.5 *[52] Given two Fuz. Nums. $\mathcal{L}, \mathcal{K}$ with θ-cuts $[\mathcal{L}]_\theta = [\mathcal{L}_{l,\theta}, \mathcal{L}_{r,\theta}]$ and $[\mathcal{K}]_\theta = [\mathcal{K}_{l,\theta}, \mathcal{K}_{r,\theta}]$ for $\theta \in]0,1]$, the norm of $\mathcal{L}$ is defined as*

$$\|\mathcal{L}\| = \sup_{\theta\in]0,1]} \max\{|\mathcal{L}_{l,\theta}|, |\mathcal{L}_{r,\theta}|\}.$$

The distance between $\mathcal{L}$ and $\mathcal{K}$ is then given by

$$D(\mathcal{L}, \mathcal{K}) = \sup_{\theta\in]0,1]} \max\{|\mathcal{L}_{l,\theta} - \mathcal{K}_{l,\theta}|, |\mathcal{L}_{r,\theta} - \mathcal{K}_{r,\theta}|\}.$$

If y is a Fuz. Num. and (y_n) is a sequence of Pos. Fuz. Nums., then

$$\lim_{n\to\infty} y_n = y \quad \Longleftrightarrow \quad \lim_{n\to\infty} D(y_n, y) = 0.$$

Definition 1.4.6 [65] *A sequence $\{y_n\}$ is called a positive solution of (2.1) if each y_n is a Pos. Fuz. Num. that satisfies (2.1).*

Moreover, $\{y_n\}$ is said to be persistent *(respectively* bounded*) if there exists a constant $\mathcal{M} > 0$ (respectively $\mathcal{N} > 0$) such that*

$$\operatorname{supp}(y_n) \subset [\mathcal{M}, +\infty[\quad (\textit{respectively } \operatorname{supp}(y_n) \subset]0, \mathcal{N}]), \quad n = 1, 2, \ldots.$$

If there exist $\mathcal{M}, \mathcal{N} \in]0, +\infty[$ such that

$$\operatorname{supp}(y_n) \subset [\mathcal{M}, \mathcal{N}], \quad n = 1, 2, \ldots,$$

then the sequence $\{y_n\}$ is both bounded and persistent.

Lemma 1.4.1 *[17] Let f be a continuous mapping from $(\mathbb{R}^+)^3$ into $\mathbb{R}^+$. If α, μ, ν are Fuz. Nums., then for all $\theta \in [0,1]$:*

$$[f(\alpha, \mu, \nu)]_\theta = f([\alpha]_\theta, [\mu]_\theta, [\nu]_\theta).$$

Theorem 1.4.1 *[8] (Stacking Theorem) If $\mathcal{L} \in \mathbb{R}_{\mathcal{F}}$ is a Fuz. Num. and $[\mathcal{L}]_\theta$ denotes its θ-cuts for $\theta \in [0,1]$, then:*

(i) Each θ-cut is a closed interval $[\mathcal{L}_{l,\theta}, \mathcal{L}_{r,\theta}]$;

(ii) If $0 \leq \theta_1 \leq \theta_2 \leq 1$, then $[\mathcal{L}]_{\theta_2} \subseteq [\mathcal{L}]_{\theta_1}$;

(iii) For any increasing sequence $\theta_n \to \theta \in]0,1]$, one has

$$\bigcap_{n=1}^{\infty} [\mathcal{L}]_{\theta_n} = [\mathcal{L}]_\theta;$$

(iv) For any decreasing sequence $\theta_n \to 0$, one has

$$\overline{\bigcup_{n=1}^{\infty} [\mathcal{L}]_{\theta_n}} = [\mathcal{L}]_0.$$

Theorem 1.4.2 [8] *Consider the functions*

$$\mathcal{L}_{l,\theta},\ \mathcal{L}_{r,\theta} : [0,1] \to \mathbb{R},$$

that satisfy the following conditions:

(i) $\mathcal{L}_{l,\theta}$ is nondecreasing, bounded, left-continuous on $]0,1]$, and right-continuous at 0;

(ii) $\mathcal{L}_{r,\theta}$ is nonincreasing, bounded, left-continuous on $]0,1]$, and right-continuous at 0;

(iii) $\mathcal{L}_{l,1} \leq \mathcal{L}_{r,1}$.

Then there exists a Fuz. Num. $\mathcal{L} \in \mathbb{R}_{\mathcal{F}}$ whose θ-cuts are given by $[\mathcal{L}_{l,\theta}, \mathcal{L}_{r,\theta}]$. Conversely, if $\mathcal{L} \in \mathbb{R}_{\mathcal{F}}$ has θ-cut endpoints $\mathcal{L}_{l,\theta}, \mathcal{L}_{r,\theta}$, then conditions (i)-(iii) are satisfied.

Definition 1.4.7 [67] *Let $\mathcal{U},\mathcal{V}\in\mathbb{R}_{\mathcal{F}}$, with $[\mathcal{U}]_\theta=[\mathcal{U}_{l,\theta},\mathcal{U}_{r,\theta}]$, $[\mathcal{V}]_\theta=[\mathcal{V}_{l,\theta},\mathcal{V}_{r,\theta}]$, and assume $0\notin[\mathcal{V}]_\theta$ for all $\theta\in[0,1]$.*

The g-division $\div_g$ is defined by $\mathcal{W}=\mathcal{U}\div_g\mathcal{V}$, where $[\mathcal{W}]_\theta=[\mathcal{W}_{l,\theta},\mathcal{W}_{r,\theta}]$ and

$$[\mathcal{W}]_\theta^{-1}=\left[\frac{1}{\mathcal{W}_{r,\theta}},\frac{1}{\mathcal{W}_{l,\theta}}\right],$$

such that

$$[\mathcal{W}]_\theta=[\mathcal{U}]_\theta\div_g[\mathcal{V}]_\theta\Longleftrightarrow\begin{cases}(i)\ [\mathcal{U}]_\theta=[\mathcal{V}]_\theta[\mathcal{W}]_\theta,\\ (ii)\ [\mathcal{V}]_\theta=[\mathcal{U}]_\theta[\mathcal{W}]_\theta^{-1}.\end{cases}$$

Remark 1.4.1 [67] *Let $\mathcal{U},\mathcal{V}\in\mathbb{R}_{\mathcal{F}}$. If $\mathcal{U}\div_g\mathcal{V}=\mathcal{W}\in\mathbb{R}_{\mathcal{F}}^+$ exists, then two possibilities occur:*

Case (i). If $\mathcal{U}_{l,\theta}\mathcal{V}_{r,\theta}\le\mathcal{U}_{r,\theta}\mathcal{V}_{l,\theta}$ for all $\theta\in[0,1]$, then

$$\mathcal{W}_{l,\theta}=\frac{\mathcal{U}_{l,\theta}}{\mathcal{V}_{l,\theta}},\quad\mathcal{W}_{r,\theta}=\frac{\mathcal{U}_{r,\theta}}{\mathcal{V}_{r,\theta}}.$$

Case (ii). If $\mathcal{U}_{l,\theta}\mathcal{V}_{r,\theta}\ge\mathcal{U}_{r,\theta}\mathcal{V}_{l,\theta}$ for all $\theta\in[0,1]$, then

$$\mathcal{W}_{l,\theta}=\frac{\mathcal{U}_{r,\theta}}{\mathcal{V}_{r,\theta}},\quad\mathcal{W}_{r,\theta}=\frac{\mathcal{U}_{l,\theta}}{\mathcal{V}_{l,\theta}}.$$

1.5 Concluding Remarks of Chapter 1

Chapter 1 has provided the fundamental mathematical framework required for the qualitative analysis developed throughout this dissertation. The concepts, definitions, and preliminary results presented here serve as the backbone for all subsequent chapters, where nonlinear difference equations and fuzzy difference equations are investigated in depth. The importance of this chapter lies not only in introducing the basic terminology and tools but also in establishing the essential theoretical foundations that enable rigorous analysis of more complex systems.

The first part of the chapter focused on nonlinear scalar difference equations, highlighting the role of equilibrium points and the various notions of stability. These concepts play a central role in understanding the long-term dynamics of discrete systems, especially when closed-form solutions are difficult or impossible to obtain. Through linearization, characteristic polynomials, and several classical stability results, the chapter provided the necessary analytical instruments that later become crucial when studying the behavior of higher-order nonlinear fuzzy models. The ideas of local and global stability, monotonicity conditions, and invariant intervals directly support the developments in Chapters 2 and 3, where fuzzy analogues of these notions are examined.

The second part of the chapter extended the discussion to systems of nonlinear difference equations. Multidimensional systems naturally arise when modeling interdependent processes, and understanding their dynamics requires tools such as the Jacobian matrix, eigenvalue analysis, and stability criteria for vector-valued iterations. These notions are indispensable for Chapter 4, where a three-dimensional nonlinear rational system is studied. In particular, the construction of the associated linearized system and the use of the spectral radius of the Jacobian matrix form the basis of the local stability analysis carried out in the final chapter.

A significant portion of the chapter was dedicated to fuzzy difference equations. The introduction of fuzzy numbers, their θ-cuts, the associated metric structure, and the arithmetic operations-including the 1-division of fuzzy numbers-constitutes a core element of the thesis. These notions enable the rigorous formulation of fuzzy difference equations and allow us to discuss existence, uniqueness, boundedness, persistence, and convergence of fuzzy solutions. Without the material provided here, the qualitative study of higher-order fuzzy equations in Chapters 2 and 3 would not be possible. In particular, the Stacking Theorem, properties of θ-cuts, and interval arithmetic play a key role in transforming a fuzzy recursive relation into a system of crisp interval relations that can be treated through classical techniques.

Overall, Chapter 1 lays the essential groundwork for the entire dissertation. Its role may be summarized as follows:

- It establishes the definitions and analytical tools that are fundamental for the qualitative study of discrete dynamical systems.
- It provides a unified language and notation system used throughout the thesis.
- It bridges the gap between classical nonlinear difference equations and their fuzzy generalizations, which are the main object of study.
- It introduces stability theory-both scalar and multidimensional which becomes indispensable in the investigation of the asymptotic behavior of the systems examined in later chapters.
- It introduces fuzzy arithmetic, fuzzy norms, and the g-division operation, which are crucial innovations employed in proving the main theorems of the thesis.

In conclusion, this chapter serves as the theoretical foundation on which all subsequent chapters are built. The qualitative investigations in Chapters 2 and 3 rely heavily on the fuzzy theoretical preliminaries established here, while the analysis in Chapter 4 depends fundamentally on the general theory of nonlinear systems and stability methods presented in this chapter. Thus, Chapter 1 is not merely introductory; it is a structural component of the dissertation, providing the analytical infrastructure necessary for the rigorous study of nonlinear fuzzy and crisp difference equations that follows.

Chapter 2

Analysis of the overall dynamics of higher-order fuzzy difference equations involving a quadratic component

2.1 Introduction

The study of systems of Fuz. Diff. Eqs. has recently become an active field of research. A Fuz. Diff. Eq. is a discrete equation in which both the coefficients and the initial conditions are expressed as Fuz. Nums., and the resulting solutions form sequences of Fuz. Nums.. This formulation is particularly valuable for describing systems influenced by uncertainty, which makes it a topic of growing theoretical interest. Over the last decade, numerous works have highlighted the relevance of Fuz. Diff. Eqs. (see [4], [11], [14], [45], [59]-[60], [74]-[79]). For example, Chrysafis et al. [11] applied them in the context of finance, suggesting an alternative methodology for studying the time value of money. Other contributions have shown strong connections between Fuz. Diff. Eqs. and fuzzy time series, fuzzy differential equations, as well as stochastic fuzzy differential equations, thereby offering a comprehensive framework for analyzing the dynamics of uncertain systems.

However, solving such equations poses challenges, mainly due to the absence of explicit solution techniques and the complexity of fuzzy arithmetic. For this reason, much of the literature focuses on the qualitative analysis of solutions, including properties such as existence, uniqueness, oscillation, and stability. This line of study has proven especially useful in understanding the long-term behavior of fuzzy dynamical models.

Motivated by these considerations, the present study is dedicated to examining the dynamics of Pos. Fuz. Sols. of the following equation:

$$y_{n+1} = \beta + \frac{\gamma y_n}{y_{n-\kappa}^2}, \quad n = 0, 1, \ldots \tag{2.1}$$

where β, γ, and $y_{-\kappa}, y_{-\kappa+1}, \ldots, y_0$, with $\kappa \in \{2, 3, \ldots\}$, are Pos. Fuz. Nums.. The move from second-order to higher-order fuzzy equations is motivated by the need to represent more complex processes, enrich the theoretical framework, and broaden the range of practical applications. This generalization strengthens our ability to investigate the qualitative behavior of systems driven by uncertainty and long-term interactions.

In addition, this study introduces the concept of g-division for Fuz. Nums., which serves as a key tool for establishing new results on the existence, uniqueness, and stability of Pos. Fuz. Sols. in higher-order Fuz. Diff. Eqs..

2.2 Conditions ensuring the existence and uniqueness of Pos. Fuz. Sols.

As this equation constitutes the starting point for exploring all possible dynamical behaviors of the system, determining the existence of a positive solution becomes a fundamental issue. The positivity of solutions is particularly significant in many real-world applications where the variables involved represent quantities that cannot take negative values, such as population sizes, concentrations, or energy levels. Establishing the existence of a positive solution ensures that the model remains meaningful and consistent with the physical or biological context it is intended to describe. Therefore, our initial objective is to demonstrate that, for any arbitrarily chosen positive initial condition, the Fuz. Diff. Eq. (2.1) admits a unique positive solution. Proving this result not only validates the internal coherence of the model but also provides a solid theoretical foundation for subsequent investigations into the system's qualitative behavior, including its stability, boundedness, and asymptotic properties.

Theorem 2.2.1 *Consider the Fuz. Diff. Eq.* (2.1)*, where* $\beta, \gamma \in \mathbb{R}^+_{\mathcal{F}}$*,* $y_{-\kappa}, y_{-\kappa+1}, \ldots, y_0 \in \mathbb{R}^+_{\mathcal{F}}$*, and* $\kappa \in \{2, 3, \ldots\}$*. Then, the equation* (2.1) *admits a* ***unique Pos. Fuz. Sol.****.*

Proof. 1. Existence of the solution

We begin by working with the θ-cuts, where $\theta \in (0, 1]$ and $n \in \mathbb{N}_0$. For this purpose, we introduce the notation:

$$\begin{cases} [y_n]_\theta = [L_{n,\theta}, R_{n,\theta}], \\ [\beta]_\theta = [\beta_{l,\theta}, \beta_{r,\theta}], \\ [\gamma]_\theta = [\gamma_{l,\theta}, \gamma_{r,\theta}]. \end{cases} \tag{2.2}$$

By applying Lemma (1.4.1) and using equations (2.1) and (2.2), we obtain:

$$\begin{aligned}
[y_{n+1}]_\theta &= [L_{n+1,\theta}, R_{n+1,\theta}] \\
&= \left[\beta + \frac{\gamma y_n}{y_{n-\kappa}^2}\right]_\theta \\
&= [\beta]_\theta + \frac{[\gamma]_\theta \times [y_n]_\theta}{[y_{n-\kappa}^2]_\theta} \\
&= [\beta_{l,\theta}, \beta_{r,\theta}] + \frac{[\gamma_{l,\theta} L_{n,\theta}, \gamma_{r,\theta} R_{n,\theta}]}{\left[L_{n-\kappa,\theta}^2, R_{n-\kappa,\theta}^2\right]}.
\end{aligned}$$

By using the g-division of Fuz. Nums., two possible cases arise:

Case (i)

$$[y_{n+1}]_\theta = \left[\beta_{l,\theta} + \frac{\gamma_{l,\theta} L_{n,\theta}}{L_{n-\kappa,\theta}^2},\ \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_{n,\theta}}{R_{n-\kappa,\theta}^2}\right]. \tag{2.3}$$

Case (ii)

$$[y_{n+1}]_\theta = \left[\beta_{l,\theta} + \frac{\gamma_{r,\theta} R_{n,\theta}}{R_{n-\kappa,\theta}^2},\ \beta_{r,\theta} + \frac{\gamma_{l,\theta} L_{n,\theta}}{L_{n-\kappa,\theta}^2}\right]. \tag{2.4}$$

Assuming Case (i) holds, we obtain:

$$L_{n+1,\theta} = \beta_{l,\theta} + \frac{\gamma_{l,\theta} L_{n,\theta}}{L_{n-\kappa,\theta}^2}, \quad R_{n+1,\theta} = \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_{n,\theta}}{R_{n-\kappa,\theta}^2}. \tag{2.5}$$

Clearly, there exists a unique solution $(L_{n,\theta}, R_{n,\theta})$ for any given initial conditions $(L_{j,\theta}, R_{j,\theta})$, with $j = -\kappa, -\kappa+1, \ldots, 0$ and $\theta \in]0,1]$. Hence:

$$[y_n]_\theta = [L_{n,\theta}, R_{n,\theta}]\,. \tag{2.6}$$

Since y_j, $j = -\kappa, \ldots, 0$, are Pos. Fuz. Nums., then by Theorem (1.4.1), for any $\theta_1, \theta_2 \in]0,1]$ such that $\theta_1 \le \theta_2$, we have:

$$0 < L_{j,\theta_1} \le L_{j,\theta_2} \le R_{j,\theta_2} \le R_{j,\theta_1}. \tag{2.7}$$

We now prove by induction that:

$$0 < L_{n,\theta_1} \leq L_{n,\theta_2} \leq R_{n,\theta_2} \leq R_{n,\theta_1}. \tag{2.8}$$

From (2.2), the statement holds for $n = -\kappa, \dots, 0$. Assume that (2.8) is valid for all $n \leq m$, with $m \in \mathbb{N}_0$. For $n = m+1$, we get:

$$\begin{aligned} L_{m+1,\theta_1} = \beta_{l,\theta_1} + \frac{\gamma_{l,\theta_1} L_{m,\theta_1}}{L^2_{m-\kappa,\theta_1}} \leq \beta_{l,\theta_2} + \frac{\gamma_{l,\theta_2} L_{m,\theta_2}}{L^2_{m-\kappa,\theta_2}} = L_{m+1,\theta_2} \\ \leq \beta_{r,\theta_2} + \frac{\gamma_{r,\theta_2} R_{m,\theta_2}}{R^2_{m-\kappa,\theta_2}} = R_{m+1,\theta_2} \leq \beta_{r,\theta_1} + \frac{\gamma_{r,\theta_1} R_{m,\theta_1}}{R^2_{m-\kappa,\theta_1}} = R_{m+1,\theta_1}, \end{aligned}$$

thus,

$$L_{m+1,\theta_1} \leq L_{m+1,\theta_2} \leq R_{m+1,\theta_2} \leq R_{m+1,\theta_1}. \tag{2.9}$$

This proves that (2.8) holds for all $n \in \mathbb{N}_0$. Moreover, from (2.5), we have:

$$L_{1,\theta} = \beta_{l,\theta} + \frac{\gamma_{l,\theta} L_{0,\theta}}{L^2_{-\kappa,\theta}}, \quad R_{1,\theta} = \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_{0,\theta}}{R^2_{-\kappa,\theta}}. \tag{2.10}$$

- According to Theorem (1.4.2), since $y_j \in \mathbb{R}^+_{\mathcal{F}}$ for $j = -\kappa, \dots, 0$ and $\beta, \gamma \in \mathbb{R}^+_{\mathcal{F}}$, it follows that $L_{j,\theta}$ and $R_{j,\theta}$ are left-continuous. Hence, $L_{1,\theta}$ and $R_{1,\theta}$ also inherit this property. By induction, the same holds for all $n \geq 1$.
- From relation (2.9), it is clear that L_θ is monotone non-decreasing, whereas R_θ is monotone non-increasing.
- Theorem (1.4.2) also ensures that both $L_{n,\theta}$ and $R_{n,\theta}$ are right-continuous at 0.
- Combining relations (2.9) and (2.10), we deduce that $L_1 \leq R_1$.

Therefore, all the requirements of Theorem (1.4.2) are fulfilled. Consequently, there exists a Pos. Fuz. Num. $y_n \in \mathbb{R}^+_{\mathcal{F}}$ such that

$$[y_n]_\theta = [L_{n,\theta}, R_{n,\theta}].$$

2. Positivity of the solution

To confirm that the obtained solution is indeed a Pos. Fuz. Num., it suffices to verify that its support

$$\operatorname{supp} y_n = \overline{\bigcup_{\theta\in]0,1]} [L_{n,\theta}, R_{n,\theta}]}$$

is compact.

Since y_j for $j = -\kappa, \dots, 0$ together with β and γ are assumed to be Pos. Fuz. Nums., there exist positive constants $M_\beta, N_\beta, M_\gamma, N_\gamma$ and M_j, N_j such that for every $\theta \in]0,1]$:

$$\begin{cases} [\beta_{l,\theta}, \beta_{r,\theta}] \subset [M_\beta, N_\beta], \\ [\gamma_{l,\theta}, \gamma_{r,\theta}] \subset [M_\gamma, N_\gamma], \\ \left[L_{j,\theta}, R_{j,\theta}\right] \subset [M_j, N_j]. \end{cases} \tag{2.11}$$

From relations (2.10) and (2.11), it follows that

$$[L_{1,\theta}, R_{1,\theta}] \subset \left[M_\beta + \frac{M_\gamma M_0}{M_{-\kappa}^2},\ N_\beta + \frac{N_\gamma N_0}{N_{-\kappa}^2}\right]. \tag{2.12}$$

Therefore,

$$\bigcup_{\theta\in]0,1]} [L_{1,\theta}, R_{1,\theta}] \subset \left[M_\beta + \frac{M_\gamma M_0}{M_{-\kappa}^2},\ N_\beta + \frac{N_\gamma N_0}{N_{-\kappa}^2}\right], \tag{2.13}$$

which shows that $\overline{\bigcup_{\theta\in]0,1]} [L_{1,\theta}, R_{1,\theta}]}$ is compact and contained in $]0, +\infty[$. By induction, the same argument applies for all n, yielding

$$\overline{\bigcup_{\theta\in]0,1]} [L_{n,\theta}, R_{n,\theta}]} \subset]0, +\infty[. \tag{2.14}$$

Hence, $[y_n]_\theta = [L_{n,\theta}, R_{n,\theta}]$ defines a sequence of Pos. Fuz. Nums..

3. Uniqueness of the solution

Suppose now that there exists another sequence $\overline{y_n}$ satisfying equation (2.1) with the same initial values $y_j \in \mathbb{R}^+_{\mathcal{F}}$ for $j = -\kappa, \dots, 0$. Using the same reasoning as before, we have

$$[\overline{y_n}]_\theta = [L_{n,\theta}, R_{n,\theta}], \quad \forall \theta \in]0,1],$$

which directly implies

$$[\overline{y_n}]_\theta = [y_n]_\theta, \quad \forall \theta \in]0,1].$$

Therefore, $\overline{y_n} = y_n$ for all $n \in \mathbb{N}_1$, and the solution is unique. ■

2.3 Investigation of bounded positive solutions of the Fuz. Diff. Eq. (2.1)

The analysis of bounded positive solutions plays a central role in understanding the long-term dynamics of the Fuz. Diff. Eq. (2.1). Once the existence and uniqueness of positive solutions have been established, it becomes essential to determine whether these solutions remain bounded for all future iterations. Boundedness ensures that the system?s trajectories do not diverge to infinity, thereby maintaining the model?s physical relevance and mathematical stability. In the context of fuzzy systems, this property acquires additional significance, as the uncertainty represented by fuzzy variables must evolve within a realistic and controlled range. By deriving appropriate inequalities and employing comparison techniques, we can establish sufficient conditions guaranteeing that every positive solution remains bounded. This analysis not only provides deeper insight into the qualitative nature of the equation but also forms a crucial step toward the study of its stability and asymptotic behavior.

2.3.1 Case (i)

In this situation, we establish the following lemma and theorem.

Lemma 2.3.1 *Consider the difference equation*

$$u_{n+1} = d + \frac{cu_n}{u_{n-\kappa}^2}, \quad n = 0, 1, \ldots, \tag{2.15}$$

where $d > 0$, $c > 0$, and the initial conditions satisfy $u_i \in]0, +\infty[$ for every $i = -\kappa, -\kappa+1, \ldots, 0$. Then, for all $n \geq 2\kappa + 2$, the sequence $\{u_n\}$ fulfills the inequality

$$d < u_n < d + \frac{c}{d}\left(1 + \frac{c}{d^2}\right)^{\kappa}. \tag{2.16}$$

Proof. We introduce the following change of variables

$$v_n = \frac{u_n}{d},$$

so that

$$u_n = dv_n,$$

thus,

$$dv_{n+1} = d + c \cdot \frac{dv_n}{d^2 v_{n-\kappa}^2},$$

simplifying,

$$\begin{aligned} dv_{n+1} &= d + \frac{c}{d} \cdot \frac{v_n}{v_{n-\kappa}^2}, \\ &= d\left(1 + \frac{c}{d^2} \cdot \frac{v_n}{v_{n-\kappa}^2}\right), \end{aligned}$$

It follows that

$$v_{n+1} = 1 + p \cdot \frac{v_n}{v_{n-\kappa}^2}, \tag{2.17}$$

where $p = \frac{c}{d^2} > 0$ and $v_i \in]0, +\infty[$ for $i = -\kappa, -\kappa + 1, \ldots, 0$.

Let $\{v_n\}_{n=-\kappa}^{\infty}$ denote a positive solution of (2.17). From (2.17), we obtain

$$v_1 = 1 + p \cdot \frac{v_0}{v_{-\kappa}^2} > 1 \quad \text{and} \quad v_2 = 1 + p \cdot \frac{v_1}{v_{1-\kappa}^2} > 1.$$

By induction, it follows that $v_n > 1$ for all $n \geq 1$.

Next, let us examine the upper bound. From (2.17), one can write

$$\begin{aligned} \frac{v_{n-1}}{v_{n-\kappa-1}^2} &= \frac{v_{n-1}}{v_{n-2}} \cdot \frac{v_{n-2}}{v_{n-\kappa-1}^2} \\ &= \frac{v_{n-1}}{v_{n-2}} \cdot \frac{v_{n-2}}{v_{n-3}} \cdot \frac{v_{n-3}}{v_{n-\kappa-1}^2} \\ &\vdots \\ &= \frac{v_{n-1}}{v_{n-2}} \cdot \frac{v_{n-2}}{v_{n-3}} \cdots \frac{v_{n-\kappa}}{v_{n-\kappa-1}^2}. \end{aligned}$$

Hence,

$$v_n = 1 + \frac{p}{v_{n-\kappa-1}} \left(\frac{v_{n-1}}{v_{n-2}} \cdot \frac{v_{n-2}}{v_{n-3}} \cdots \frac{v_{n-\kappa}}{v_{n-\kappa-1}} \right). \tag{2.18}$$

Since $v_n > 1$ for all $n \geq 1$, we have

$$v_{n-\kappa-1} > 1,$$

which implies

$$\frac{1}{v_{n-\kappa-1}} < 1.$$

Therefore,

$$\frac{p}{v_{n-\kappa-1}} < p,$$

and multiplying both sides by the above product yields the desired inequality.

$$\frac{p}{v_{n-\kappa-1}} \left(\frac{v_{n-1}}{v_{n-2}} \cdots \frac{v_{n-\kappa}}{v_{n-\kappa-1}} \right) < p \left(\frac{v_{n-1}}{v_{n-2}} \cdots \frac{v_{n-\kappa}}{v_{n-\kappa-1}} \right),$$

hence,

$$v_n < 1 + p\left(\frac{v_{n-1}}{v_{n-2}} \cdot \frac{v_{n-2}}{v_{n-3}} \cdots \frac{v_{n-\kappa}}{v_{n-\kappa-1}}\right),$$

so we obtain

$$v_n < 1 + p\left(\frac{v_{n-1}}{v_{n-2}} \cdot \frac{v_{n-2}}{v_{n-3}} \cdots \frac{v_{n-\kappa}}{v_{n-\kappa-1}}\right). \tag{2.19}$$

Additionally, from (2.17) and for i = 1, 2, . . .

$$v_i = 1 + p \cdot \frac{v_{i-1}}{v^2_{i-\kappa-1}} = \frac{v_{i-1}}{v_{i-1}} + p \cdot \frac{v_{i-1}}{v^2_{i-\kappa-1}},$$

hence,

$$v_i = v_{i-1}\left(\frac{1}{v_{i-1}} + \frac{p}{v^2_{i-\kappa-1}}\right),$$

as a result,

$$\frac{v_i}{v_{i-1}} = \frac{1}{v_{i-1}} + \frac{p}{v^2_{i-\kappa-1}},$$

thus, from (2.18) and (2.19):

$$v_n < 1 + p\left(\frac{1}{v_{n-2}} + \frac{p}{v^2_{n-\kappa-2}}\right) \cdot \left(\frac{1}{v_{n-3}} + \frac{p}{v^2_{n-\kappa-3}}\right) \cdots \left(\frac{1}{v_{n-\kappa-1}} + \frac{p}{v^2_{n-2\kappa-1}}\right). \tag{2.20}$$

If $n \geq 2\kappa + 2$, then $v_{n-i} > 1$ for $i = 2, \ldots, 2\kappa + 1$, so

$$\frac{1}{v_{n-i}} < 1, \quad \frac{1}{v^2_{n-\kappa-i}} < 1, \quad \frac{p}{v^2_{n-\kappa-i}} < p,$$

then,

$$\frac{1}{v_{n-i}} + \frac{p}{v^2_{n-\kappa-i}} < 1 + p, \tag{2.21}$$

from (2.20) and (2.21), we obtain:

$$v_n < 1 + p(1 + p)^{\kappa},$$

therefore, for $n \geq 2\kappa + 2$

$$d < u_n < d + \frac{c}{d}\left(1 + \frac{c}{d^2}\right)^{\kappa}.$$

Hence, the proof is established. ■

Theorem 2.3.1 *Let us consider the Fuz. Diff. Eq.* (2.1)*. Assume that there exist constants* $M_\beta, N_\beta, M_\gamma, N_\gamma > 0$ *such that, for all* $\theta \in]0,1]$*, the inequalities*

$$\begin{cases} M_\beta \leq \beta_{l,\theta} \leq \beta_{r,\theta} \leq N_\beta, \\ M_\gamma \leq \gamma_{l,\theta} \leq \gamma_{r,\theta} \leq N_\gamma, \end{cases} \tag{2.22}$$

hold. Then every Pos. Fuz. Sol. y_n *of the Fuz. Diff. Eq.* (2.1) *is bounded and persistent.*

Proof. Let y_n denote a Pos. Fuz. Sol. of the Fuz. Diff. Eq. (2.1). From relation (2.5), we can immediately deduce that

$$\begin{cases} M_\beta \leq \beta_{l,\theta} \leq L_{n,\theta} \text{ which implies that } M_\beta \leq L_{n,\theta}, \\ M_\beta \leq \beta_{r,\theta} \leq R_{n,\theta} \text{ which implies that } M_\beta \leq R_{n,\theta}, \end{cases} \tag{2.23}$$

From (2.5) and (2.15), if we set $\beta_{r,\theta} = d$, $\gamma_{r,\theta} = c$, and $R_n = u_n$, then from (2.16) we obtain

$$R_n < \beta_{r,\theta} + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}}\left(1 + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}^2}\right)^{\kappa}. \tag{2.24}$$

From (2.23) we have

$$\begin{cases} \beta_{r,\theta} \leq N_\beta, \\ \beta_{r,\theta} \geq M_\beta \text{ thus } \frac{1}{\beta_{r,\theta}} \leq \frac{1}{M_\beta}, \\ \beta_{r,\theta}^2 \geq M_\beta^2 \text{ thus } \frac{1}{\beta_{r,\theta}^2} \leq \frac{1}{M_\beta^2}, \\ \gamma_{r,\theta} \leq N_\gamma. \end{cases}$$

Thus,

$$\frac{\gamma_{r,\theta}}{\beta_{r,\theta}^2} \leq \frac{N_\gamma}{M_\beta^2},$$

and consequently,

$$1 + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}^2} \leq 1 + \frac{N_\gamma}{M_\beta^2}.$$

Moreover,

$$\left(1 + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}^2}\right)^\kappa \leq \left(1 + \frac{N_\gamma}{M_\beta^2}\right)^\kappa,$$

which implies that

$$\frac{\gamma_{r,\theta}}{\beta_{r,\theta}}\left(1 + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}^2}\right)^\kappa \leq \frac{N_\gamma}{M_\beta}\left(1 + \frac{N_\gamma}{M_\beta^2}\right)^\kappa.$$

Therefore,

$$\beta_{r,\theta} + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}}\left(1 + \frac{\gamma_{r,\theta}}{\beta_{r,\theta}^2}\right)^\kappa \leq N_\beta + \frac{N_\gamma}{M_\beta}\left(1 + \frac{N_\gamma}{M_\beta^2}\right)^\kappa.$$

We thus have

$$R_n \leq N_\beta + \frac{N_\gamma}{M_\beta}\left(1 + \frac{N_\gamma}{M_\beta^2}\right)^\kappa. \tag{2.25}$$

From (2.23) and (2.25) it follows that

$$M_\beta \leq L_{n,\theta} \leq R_{n,\theta} \leq N_\beta + \frac{N_\gamma}{M_\beta}\left(1 + \frac{N_\gamma}{M_\beta^2}\right)^\kappa. \tag{2.26}$$

For $n \leq 2\kappa + 2$ we have

$$[L_{n,\theta}, R_{n,\theta}] \subset \left[M_\beta, T\right], \tag{2.27}$$

where

$$T = N_\beta + \frac{N_\gamma}{M_\beta}\left(1 + \frac{N_\gamma}{M_\beta^2}\right)^\kappa.$$

Consequently, (2.27) implies that for $n \geq 2\kappa + 2$,

$$\bigcup_{\theta \in]0,1]} [L_{n,\theta}, R_{n,\theta}] \subset \left[M_\beta, T\right].$$

Hence,

$$\overline{\bigcup_{\theta\in]0,1]}[L_{n,\theta},R_{n,\theta}]}\subseteq\left[M_{\beta},T\right].$$

Consequently, y_n, being a Pos. Fuz. Sol., possesses both boundedness and persistence.

■

2.3.2 Case(ii)

When case (ii) is valid, the following Lemma and Theorem become essential.

Lemma 2.3.2 *Consider the system of difference equations defined below:*

$$w_{n+1}=p+\frac{d\,z_n}{z_{n-\kappa}^2},\quad z_{n+1}=q+\frac{c\,w_n}{w_{n-\kappa}^2},\quad n=0,1,\ldots,\tag{2.28}$$

such that w_i, z_i are positive real numbers for every $i=-\kappa,-\kappa+1,\ldots,0$. If

$$p^2>d>1,\quad q^2>c>1,\tag{2.29}$$

Consequently, for all $n\geq 2\kappa+2$,

$$p\leq w_n\leq\frac{p\,(p^2q^2+pqd)}{p^2q^2-cd}+w_{\kappa+1},\quad q\leq z_n\leq\frac{q\,(p^2q^2+pqc)}{p^2q^2-cd}+z_{\kappa+1}.\tag{2.30}$$

Proof. From the system of difference equations (2.28), we obtain

$$w_{n+1}=p\left(1+\frac{d}{p}\frac{z_n}{z_{n-\kappa}^2}\right),\quad z_{n+1}=q\left(1+\frac{c}{q}\frac{w_n}{w_{n-\kappa}^2}\right),\quad n=0,1,\ldots,$$

then

$$\frac{w_{n+1}}{p}=1+\frac{d}{p}\frac{z_n}{z_{n-\kappa}^2},\quad \frac{z_{n+1}}{q}=1+\frac{c}{q}\frac{w_n}{w_{n-\kappa}^2},\quad n=0,1,\ldots,$$

Let $W_n = \frac{w_n}{p}$ and $Z_n = \frac{z_n}{q}$, so $w_n = pW_n$ and $z_n = qZ_n$. Then, we can transform equation (2.28) into the following system:

$$W_{n+1} = 1 + \frac{d}{p}\frac{qZ_n}{q^2 Z_{n-\kappa}^2}, \quad Z_{n+1} = 1 + \frac{c}{q}\frac{pW_n}{p^2 W_{n-\kappa}^2}.$$

Thus,

$$W_{n+1} = 1 + D\frac{Z_n}{Z_{n-\kappa}^2}, \quad Z_{n+1} = 1 + C\frac{W_n}{W_{n-\kappa}^2}, \quad n = 0, 1, 2, \ldots \tag{2.31}$$

where $D = \frac{d}{pq}$ and $C = \frac{c}{pq}$.

From (2.31), for $n \geq 1$, we have $W_n \geq 1$ and $Z_n \geq 1$. Moreover, for $n \geq 2\kappa + 2$,

$$Z_{n-1-\kappa} \geq 1, \quad \text{so} \quad \frac{1}{Z_{n-1-\kappa}^2} \leq 1,$$

which implies

$$W_n = 1 + D\frac{Z_{n-1}}{Z_{n-1-\kappa}^2} \leq 1 + DZ_{n-1},$$

In addition, we have

$$W_{n-2-\kappa} \geq 1, \quad \text{so} \quad \frac{1}{W_{n-2-\kappa}^2} \leq 1,$$

which implies

$$Z_{n-1} = 1 + C\frac{W_{n-2}}{W_{n-2-\kappa}^2} \leq 1 + CW_{n-2}.$$

We obtain

$$W_n \leq 1 + DZ_{n-1} \leq 1 + D + CDW_{n-2}. \tag{2.32}$$

Similarly, we have

$$W_{n-1-\kappa} \geq 1 \quad \text{so} \quad \frac{1}{W_{n-1-\kappa}^2} \leq 1,$$

which implies

$$Z_n = 1 + C\frac{W_{n-1}}{W_{n-1-\kappa}^2} \leq 1 + CW_{n-1},$$

In turn, we get

$$Z_{n-2-\kappa} \geq 1, \quad \text{so} \quad \frac{1}{Z_{n-2-\kappa}^2} \leq 1,$$

hence

$$W_{n-1} = 1 + D\frac{Z_{n-2}}{Z_{n-2-\kappa}^2} \leq 1 + DZ_{n-2}.$$

We obtain

$$Z_n \leq 1 + CW_{n-1} \leq 1 + C + CDZ_{n-2}. \tag{2.33}$$

By induction, for $n - 2m \geq \kappa + 1$, we have

$$\begin{aligned}
& W_n \leq 1 + D + CDW_{n-2} \\
\leq \;& 1 + D + CD + CD^2 Z_{n-3} \\
\leq \;& 1 + D + CD + CD^2 + C^2D^2 W_{n-4} \\
\vdots \;& \\
\leq \;& \sum_{i=0}^{m-1}(CD)^i + D\sum_{i=0}^{m-1}(CD)^i + (CD)^m W_{n-2m} \\
= \;& \frac{1-(CD)^m}{1-CD} + D\frac{1-(CD)^m}{1-CD} + (CD)^m W_{n-2m}.
\end{aligned}$$

From (2.29), we have

$$p^2 > d \quad \text{therefore} \quad \frac{d}{p^2} < 1,$$
$$q^2 > c \quad \text{therefore} \quad \frac{c}{q^2} < 1,$$

then

$$CD = \frac{cd}{p^2q^2} < 1.$$

Thus,

$$W_n \leq \frac{1}{1-CD} + \frac{D}{1-CD} + W_{n-2m},$$
$$W_n \leq \frac{p^2q^2 + pqd}{p^2q^2 - cd} + W_{n-2m}.$$

In the same way, we find

$$\begin{aligned}
& Z_n \leq 1 + C + CDZ_{n-2} \\
\leq \ & 1 + C + CD + C^2DW_{n-3} \\
\leq \ & 1 + C + CD + C^2D + C^2D^2Z_{n-4} \\
\vdots \ & \\
\leq \ & \sum_{i=0}^{m-1}(CD)^i + C\sum_{i=0}^{m-1}(CD)^i + (CD)^m Z_{n-2m} \\
= \ & \frac{1-(CD)^m}{1-CD} + C\frac{1-(CD)^m}{1-CD} + (CD)^m Z_{n-2m},
\end{aligned}$$

which gives

$$Z_n \leq \frac{1}{1-CD} + \frac{C}{1-CD} + Z_{n-2m},$$
$$Z_n \leq \frac{p^2q^2 + pqc}{p^2q^2 - cd} + Z_{n-2m}.$$

Observing that $n - 2m \geq \kappa + 1$ can be rewritten as $m \leq \frac{n-\kappa-1}{2}$, we obtain the desired conclusion. ■

Theorem 2.3.2 *Let us consider the Fuz. Diff. Eq.* (2.1). *Assume that there exist positive constants $M_\beta, N_\beta, M_\gamma$, and N_γ such that, for every $\theta \in]0,1]$, the following conditions hold:*

$$\begin{cases} M_\beta \leq \beta_{l,\theta} \leq \beta_{r,\theta} \leq N_\beta, \\ M_\gamma \leq \gamma_{l,\theta} \leq \gamma_{r,\theta} \leq N_\gamma, \\ M_\beta^2 > N_\gamma, \end{cases} \tag{2.34}$$

Under these assumptions, each Pos. Fuz. Sol. y_n of the Fuz. Diff. Eq. (2.1) *proves to be persistent as well as bounded.*

Proof. Suppose that (y_n) is a positive solution of (2.1). Referring to (2.5) and (2.28), we have

$$\begin{aligned} p = \beta_{l,\theta} \quad &; \quad q = \beta_{r,\theta}, \\ c = \gamma_{l,\theta} \quad &; \quad d = \gamma_{r,\theta}. \end{aligned}$$

From (2.30) and (2.34), we obtain

$$\begin{cases} \beta_{l,\theta} \leq N_\beta, \\ \beta_{r,\theta} \leq N_\beta, \\ \gamma_{l,\theta} \leq N_\gamma, \end{cases} \quad \text{which implies} \quad \begin{cases} \beta_{l,\theta}\beta_{r,\theta}\gamma_{l,\theta} \leq N_\beta^2 N_\gamma, \\ \beta_{l,\theta}^2 \beta_{r,\theta}^2 \leq N_\beta^4, \end{cases}$$

$$\beta_{r,\theta}(\beta_{l,\theta}^2\beta_{r,\theta}^2 + \beta_{l,\theta}\beta_{r,\theta}\gamma_{l,\theta}) \leq N_\beta(N_\beta^4 + N_\beta^2 N_\gamma). \tag{2.35}$$

$$\begin{cases} \beta_{l,\theta} \geq M_\beta, \\ \beta_{r,\theta} \geq M_\beta, \end{cases} \quad \text{which leads to} \quad \beta_{l,\theta}^2\beta_{r,\theta}^2 \geq M_\beta^4.$$

$$\begin{cases} \gamma_{l,\theta} \leq N_{\gamma}, \\ \gamma_{r,\theta} \leq N_{\gamma}, \end{cases} \qquad \text{which gives} \qquad -\gamma_{l,\theta}\gamma_{r,\theta} \geq -N_{\gamma}^2,$$

thus,

$$\beta_{l,\theta}^2\beta_{r,\theta}^2 - \gamma_{l,\theta}\gamma_{r,\theta} \geq M_{\beta}^4 - N_{\gamma}^2,$$

and consequently

$$\frac{1}{\beta_{l,\theta}^2\beta_{r,\theta}^2 - \gamma_{l,\theta}\gamma_{r,\theta}} \leq \frac{1}{M_{\beta}^4 - N_{\gamma}^2}. \tag{2.36}$$

From (2.35) and (2.36) we deduce that

$$R_{n,\theta} \leq \frac{N_{\beta}(N_{\beta}^4 + N_{\beta}^2 N_{\gamma})}{M_{\beta}^4 - N_{\gamma}^2} + R_{\kappa+1,\theta}.$$

Therefore,

$$[L_{n,\theta}, R_{n,\theta}] \subset \left[M_{\beta}, T_{\theta}\right],$$

For $n \geq 2\kappa + 2$, let

$$T_{\theta} = \frac{N_{\beta}(N_{\beta}^4 + N_{\beta}^2 N_{\gamma})}{M_{\beta}^4 - N_{\gamma}^2} + R_{\kappa+1,\theta}.$$

Since the sequence (y_n) consists of Pos. Fuz. Nums., there exists a constant $T > 0$ such that $T_{\theta} \leq T$ for all $\theta \in]0, 1]$. Consequently, for every $n \geq 2\kappa + 2$ we obtain

$$[L_{n,\theta}, R_{n,\theta}] \subset [0, T].$$

Thus,

$$\bigcup_{\theta \in]0,1]} [L_{n,\theta}, R_{n,\theta}] \subset [0, T], \quad n \geq 2\kappa + 2.$$

By taking the closure, it follows that

$$\overline{\bigcup_{\theta \in]0,1]} [L_{n,\theta}, R_{n,\theta}]} \subseteq [0, T].$$

Hence, the proof of case (ii) is established. ■

2.4 On the convergence of solutions to the Fuz. Diff. Eq. (2.1)

We now proceed to demonstrate that the Pos. Fuz. Sol. y_n of equation (2.1) is convergent as $n \to \infty$. Establishing the convergence of y_n is a crucial step toward understanding the asymptotic behavior of the system and confirming the stability of its equilibrium point. To achieve this goal, we begin by deriving several auxiliary lemmas that serve as essential tools for the main proof. These lemmas provide intermediate results concerning the properties of the sequence (y_n), such as its monotonicity, boundedness, or contractive nature, which together form the foundation for proving convergence. Once these preliminary results are established, they will allow us to rigorously demonstrate that the sequence y_n tends to a finite limit, thereby ensuring that the Fuz. Diff. Eq. exhibits well-defined long-term behavior.

2.4.1 Case(i)

Lemma 2.4.1 *Consider the difference equation given in* (2.15).

If the condition $d^2 > \frac{4c}{3}$ holds, then the equilibrium point $\bar{u}$ of (2.15) *is locally asymptotically stable.*

Proof. We start from the equation

$$
\begin{aligned}
\bar{u} &= d + c\frac{\bar{u}}{\bar{u}^2}, \\
&= d + \frac{c}{\bar{u}}, \\
&= \frac{d\bar{u}+c}{\bar{u}},
\end{aligned}
$$

which leads to

$$
\bar{u}^2 - d\bar{u} - c = 0,
$$

$$
\Delta = d^2 + 4c > 0,
$$

therefore

$$\bar{u}_1 = \tfrac{d+\sqrt{d^2+4c}}{2}, \quad \bar{u}_2 = \tfrac{d-\sqrt{d^2+4c}}{2}.$$

Therefore, the equation (2.15) admits a single positive equilibrium given by

$$\bar{u} = \frac{d + \sqrt{d^2 + 4c}}{2}.$$

Moreover, the linearization of (2.15) in a neighborhood of this equilibrium takes the form:

$$u_{n+1} - \frac{2c}{d^2 + 2c + d\sqrt{d^2 + 4c}}\, u_n + \frac{4c}{d^2 + 2c + d\sqrt{d^2 + 4c}}\, u_{n-\kappa} = 0. \tag{2.37}$$

We introduce the following function $f : I^{\kappa+1} \longrightarrow I$ by

$$f(u_n, u_{n-1}, \ldots, u_{n-\kappa}) = d + \frac{c\, u_n}{u_{n-\kappa}^2}, \tag{2.38}$$

Here, f is assumed to be continuously differentiable, and I represents an interval contained within the set of positive real numbers.

We compute

$$\frac{\partial f}{\partial u_n}(u_n, u_{n-1}, \ldots, u_{n-\kappa}) = \frac{c}{u_{n-\kappa}^2},$$

thus

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial u_n}(\bar{u}, \bar{u}, \ldots, \bar{u}) \\ &= \frac{c}{\bar{u}^2} \\ &= \frac{c}{\left(\frac{d+\sqrt{d^2+4c}}{2}\right)^2} \\ &= \frac{2c}{d^2 + 2c + d\sqrt{d^2 + 4c}}. \end{aligned}$$

Similarly, we have

$$\frac{\partial f}{\partial u_{n-\kappa}}(u_n, u_{n-1}, \ldots, u_{n-\kappa}) \quad = \quad \frac{-2c\, u_{n-\kappa}\, u_n}{u_{n-\kappa}^4} = \frac{-2c\, u_n}{u_{n-\kappa}^3},$$

which gives

$$\begin{aligned} q_\kappa &= \frac{\partial f}{\partial u_{n-\kappa}}(\bar{u}, \bar{u}, \ldots, \bar{u}) \\ &= \frac{-2c}{\bar{u}^2} \\ &= \frac{-2c}{\left(\frac{d+\sqrt{d^2+4c}}{2}\right)^2} \\ &= \frac{-4c}{d^2 + 2c + d\sqrt{d^2+4c}}. \end{aligned}$$

Moreover, we note that

$$q_1 = q_2 = \cdots = q_{\kappa-1} = 0.$$

Therefore, the characteristic equation of (2.15) becomes

$$\lambda^{\kappa+1} - \frac{2c}{d^2 + 2c + d\sqrt{d^2+4c}}\lambda^\kappa + \frac{4c}{d^2 + 2c + d\sqrt{d^2+4c}} = 0. \tag{2.39}$$

From the expressions above, we find that

$$|q_0| + |q_\kappa| = \frac{6c}{d^2 + 2c + d\sqrt{d^2+4c}},$$

under the condition $d^2 > \frac{4}{3}c$.

From this condition, it follows that

$$d^2 + 4c > \frac{4}{3}c + 4c = \frac{16c}{3},$$

consequently,

$$\sqrt{d^2+4c} > \frac{4\sqrt{c}}{\sqrt{3}}. \tag{2.40}$$

On the other hand, we also obtain

$$d > \frac{2\sqrt{c}}{\sqrt{3}}. \tag{2.41}$$

Combining (2.40) and (2.41), we deduce

$$d\sqrt{d^2+4c} > \frac{8c}{3},$$

therefore

$$d^2 + 2c + d\sqrt{d^2+4c} > \frac{4c}{3} + 2c + \frac{8c}{3} = 6c,$$

therefore

$$\frac{6c}{d^2+2c+d\sqrt{d^2+4c}} < 1,$$

thus

$$|q_0| + |q_\kappa| < 1.$$

Therefore, the equilibrium point $\bar{u}$ of equation (2.15) is locally stable in the asymptotic sense. ■

Lemma 2.4.2 *Consider equation* (2.15). *If the condition $d^2 > 3c$ holds, then the positive* *solution* (y_n) *converges to the equilibrium point* $\bar{u}$.

Proof.

$$m, M \in \left[d,\ d + \frac{c}{d}\left(1 + \frac{c}{d^2}\right)^k\right]$$

$$m = f(m, M), \qquad M = f(M, m) \tag{2.42}$$

Assume that

$$m \leq M$$

(the case $M \leq m$ can be treated similarly).

From (2.42), we have

$$m = d + \frac{cm}{M^2}, \qquad M = d + \frac{cM}{m^2}$$

Then

$$\begin{aligned} M - m &= c\left(\frac{M}{m^2} - \frac{m}{M^2}\right) \\ &= c\left(\frac{M^3 - m^3}{m^2 M^2}\right) \\ &= \frac{c(M - m)(M^2 + mM + m^2)}{m^2 M^2} \end{aligned}$$

Since

$$mM \leq M^2, \qquad m^2 \leq M^2$$

then

$$M^2 + mM + m^2 \leq 3M^2$$

Thus

$$M - m \leq \frac{c(M-m)(3M^2)}{m^2M^2}$$

Hence

$$M - m \leq \frac{3c(M-m)}{m^2}$$

Since

$$m \geq d$$

we get

$$M - m \leq \frac{3c}{d^2}(M-m)$$

Therefore

$$\left(1 - \frac{3c}{d^2}\right)(M-m) \leq 0 \tag{2.43}$$

We want

$$M - m \leq 0$$

From the condition

$$d^2 > 3c,$$

we obtain

$$1 - \frac{3c}{d^2} > 0$$

Thus, from (2.43),

$$M - m \leq 0$$

which implies

$$M \leq m$$

Since we already assumed

$$m \leq M$$

we conclude that

$$M = m$$

Thus, convergence is established. ∎

Lemma 2.4.3 *For the difference equation* (2.15)*, if the condition* $d^2 > 3c$ *holds, then the unique positive equilibrium point* $\bar{u}$ *is globally asymptotically stable.*

Theorem 2.4.1 *Assume that the conditions*

$$\beta_{l,\theta}^2 > 3\gamma_{l,\theta}, \quad \beta_{r,\theta}^2 > 3\gamma_{r,\theta}, \quad \forall\theta \in]0,1]$$

are satisfied. Then, every Pos. Fuz. Sol. y_n *of the Fuz. Diff. Eq.* (2.1) *converges to the positive fuzzy equilibrium* y *as* $n \to \infty$.

Proof. Assume that there exists a Fuz. Num. y satisfying

$$y = \beta + \frac{\gamma y}{y^2}, \qquad [y]_\theta = [L_\theta, R_\theta], \quad \theta \in]0,1], \tag{2.44}$$

where $L_\theta, R_\theta \geq 0$. From relation (2.44), we can deduce the following:

$$L_\theta = \beta_{l,\theta} + \frac{\gamma_{l,\theta} L_\theta}{L_\theta^2}, \quad R_\theta = \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_\theta}{R_\theta^2}, \tag{2.45}$$

hence we have from (2.45) that

$$\begin{aligned} L_\theta &= \beta_{l,\theta} + \gamma_{l,\theta}\frac{L_\theta}{L_\theta^2} \\ &= \beta_{l,\theta} + \frac{\gamma_{l,\theta}}{L_\theta} \\ &= \frac{\beta_{l,\theta} L_\theta + \gamma_{l,\theta}}{L_\theta}, \end{aligned}$$

which implies

$$L_\theta^2 - \beta_{l,\theta} L_\theta - \gamma_{l,\theta} = 0,$$
$$\Delta = L_\theta^2 + 4\gamma_{l,\theta} > 0,$$

we obtain

$$L_{\delta 1} = \frac{\beta_{l,\theta} + \sqrt{\beta_{l,\theta}^2 + 4\gamma_{l,\theta}}}{2} \quad , \quad L_{\delta 2} = \frac{\beta_{l,\theta} - \sqrt{\beta_{l,\theta}^2 + 4\gamma_{l,\theta}}}{2}.$$

Likewise

$$\begin{aligned} R_\theta &= \beta_{r,\theta} + \gamma_{r,\theta} \frac{R_\theta}{R_\theta^2} \\ &= \beta_{r,\theta} + \frac{\gamma_{r,\theta}}{R_\theta} \\ &= \frac{\beta_{r,\theta} R_\theta + \gamma_{r,\theta}}{R_\theta}, \end{aligned}$$

which implies

$$R_\theta^2 - \beta_{r,\theta} R_\theta - \gamma_{r,\theta} = 0,$$
$$\Delta = R_\theta^2 + 4\gamma_{r,\theta} > 0,$$

hence

$$R_{\delta 1} = \frac{\beta_{r,\theta} + \sqrt{\beta_{r,\theta}^2 + 4\gamma_{r,\theta}}}{2} \quad , \quad R_{\delta 2} = \frac{\beta_{r,\theta} - \sqrt{\beta_{r,\theta}^2 + 4\gamma_{r,\theta}}}{2}.$$

We have (L_θ, R_θ) is the unique equilibrium point of (2.45), such that:

$$L_\theta = \frac{\beta_{l,\theta} + \sqrt{\beta_{l,\theta}^2 + 4\gamma_{l,\theta}}}{2}, \quad R_\theta = \frac{\beta_{r,\theta} + \sqrt{\beta_{r,\theta}^2 + 4\gamma_{r,\theta}}}{2}.$$

Consider y_n as a Pos. Fuz. Sol. corresponding to the Fuz. Diff. Eq. (2.1). Since $\beta_{l,\theta}^2 > \frac{4}{3}\gamma_{l,\theta}, \beta_{r,\theta}^2 > \frac{4}{3}\gamma_{r,\theta}, \theta \in]0,1]$. Utilizing Lemma (2.4.3), then

$$\lim_{n\to\infty} L_{n,\theta} = L_\theta, \quad \lim_{n\to\infty} R_{n,\theta} = R_\theta, \tag{2.46}$$

then, from (2.46) it has

$$\lim_{n\to\infty} D(y_n, y) = \lim_{n\to\infty} \sup_{\theta\in(0,1]} \{\max\{| L_{n,\theta} - L_\theta |, | R_{n,\theta} - R_\theta |\}\} = 0.$$

This brings the demonstration of the theorem to an end. ■

2.4.2 Case(ii)

Lemma 2.4.4 *Consider the system given in* (2.28). *When condition* (2.29) *is fulfilled, the unique positive equilibrium point* $(\bar{w}, \bar{z})$ *of* (2.28) *is locally asymptotically stable.*

Proof. From (2.28), we are going to calculate the positive equilibrium point $(\bar{w}, \bar{z})$

$$\begin{aligned}
\bar{w} &= p + \frac{d\bar{z}}{\bar{z}^2} \\
&= \frac{p\bar{z} + d}{\bar{z}}, \\
\bar{z} &= \frac{q\frac{p\bar{z}+d}{\bar{z}} + c}{\frac{p\bar{z}+d}{\bar{z}}} \\
&= \frac{\frac{qp\bar{z}+qd+c\bar{z}}{\bar{z}}}{\frac{p\bar{z}+d}{\bar{z}}} \\
&= \frac{(pq+c)\bar{z} + qd}{p\bar{z} + d},
\end{aligned}$$

therfore

$$\begin{aligned}
&\bar{z}(p\bar{z} + d) = (pq + c)\bar{z} + qd, \\
&p\bar{z}^2 - (pq - d + c)\bar{z} - qd = 0, \\
&\Delta = (pq - d + c)^2 + 4pqd > 0,
\end{aligned}$$

which gives

$$\begin{cases}
\bar{z}_1 = \dfrac{pq - d + c + \sqrt{(pq - d + c)^2 + 4pqd}}{2p}, \\
\bar{z}_2 = \dfrac{pq - d + c - \sqrt{(pq - d + c)^2 + 4pqd}}{2p}.
\end{cases}$$

$$
\begin{aligned}
\bar{w} &= p + \frac{d\bar{z}}{\bar{z}^2} \\
&= p + \frac{d}{\bar{z}} \\
&= p + \frac{d}{\frac{pq-d+c+\sqrt{(pq-d+c)^2+4pqd}}{2p}} \\
&= p + \frac{2pd\left(pq-d+c-\sqrt{(pq-d+c)^2+4pqd}\right)}{(pq-d+c+\sqrt{(pq-d+c)^2+4pqd})(pq-d+c-\sqrt{(pq-d+c)^2+4pqd})} \\
&= p + \frac{2pd\left(pq-d+c-\sqrt{(pq-d+c)^2+4pqd}\right)}{(pq-d+c)^2-((pq-d+c)^2+4pqd)} \\
&= p + \frac{2pd\left(pq-d+c-\sqrt{(pq-d+c)^2+4pqd}\right)}{-4pqd} \\
&= \frac{pq-c+d+\sqrt{(pq-d+c)^2+4pqd}}{2q}.
\end{aligned}
$$

So, we obtain

$$
(\bar{w},\bar{z}) = \left(\frac{pq-c+d+\sqrt{(pq-d+c)^2+4pqd}}{2q}, \frac{pq-d+c+\sqrt{(pq-d+c)^2+4pqd}}{2p}\right).
$$

The linearization of system (2.28) around the equilibrium point $(\bar{w},\bar{z})$ is given by

$$
\Psi_{n+1} = \Gamma\Psi_n,
$$

here $\Psi_n = (w_n, w_{n-1}, \cdots, w_{n-\kappa}, z_n, z_{n-1}, \cdots, z_{n-\kappa})^T$, The system that corresponds to (2.28) can be expressed as

$$\begin{cases} w_{n+1} = p + \frac{dz_n}{z_{n-\kappa}^2}, \\ w_n = \qquad w_n, \\ \vdots \\ w_{n-\kappa+1} = \quad w_{n-\kappa+1} \\ z_{n+1} = q + \frac{cw_n}{w_{n-\kappa}^2}, \\ z_n = \qquad z_n, \\ \vdots \\ z_{n-\kappa+1} = \quad z_{n-\kappa+1}. \end{cases}$$

At first, let f and g continuously differentiable, such that:

$$f :]0, +\infty[^{2\kappa+2} \to]0, +\infty[^{\kappa+1}, \quad g :]0, +\infty[^{2\kappa+2} \to]0, +\infty[^{\kappa+1}$$

and

$$f = (f_1, f_2, \cdots, f_{\kappa+1}), \quad g = (g_1, g_2, \cdots, g_{\kappa+1})$$

$$\begin{cases} f_1 : &]0,+\infty[^{2\kappa+2} & \rightarrow &]0,+\infty[\\ (w_1, w_2, \cdots, w_{\kappa+1}, z_1, z_2, \cdots, z_{\kappa+1}) & \mapsto & p + \frac{dz_1}{z_{\kappa+1}^2} \\ f_2 : &]0,+\infty[^{2\kappa+2} & \rightarrow &]0,+\infty[\\ (w_1, w_2, \cdots, w_{\kappa+1}, z_1, z_2, \cdots, z_{\kappa+1}) & \mapsto & w_1 \\ \vdots \\ f_{\kappa+1} : &]0,+\infty[^{2\kappa+2} & \rightarrow &]0,+\infty[\\ (w_1, w_2, \cdots, w_{\kappa+1}, z_1, z_2, \cdots, z_{\kappa+1}) & \mapsto & w_\kappa \\ g_1 : &]0,+\infty[^{2\kappa+2} & \rightarrow &]0,+\infty[\\ (w_1, w_2, \cdots, w_{\kappa+1}, z_1, z_2, \cdots, z_{\kappa+1}) & \mapsto & q + \frac{cw_1}{w_{\kappa+1}^2} \\ g_2 : &]0,+\infty[^{2\kappa+2} & \rightarrow &]0,+\infty[\\ (w_1, w_2, \cdots, w_{\kappa+1}, z_1, z_2, \cdots, z_{\kappa+1}) & \mapsto & z_1 \\ \vdots \\ g_{\kappa+1} : &]0,+\infty[^{2\kappa+2} & \rightarrow &]0,+\infty[\\ (w_1, w_2, \cdots, w_{\kappa+1}, z_1, z_2, \cdots, z_{\kappa+1}) & \mapsto & z_\kappa \end{cases}$$

We present below the Jacobian matrix evaluated at the equilibrium point $\bar{W} = (\bar{w}, \bar{w}, \cdots, \bar{w}, \bar{z}, \bar{z}, \cdots, \bar{z})$

$$\Gamma_{(2\kappa+2)\times(2\kappa+2)} = \begin{pmatrix} \frac{\partial f_1}{\partial w_1}(\bar{W}) & \frac{\partial f_1}{\partial w_2}(\bar{W}) & \cdots & \frac{\partial f_1}{\partial w_{\kappa+1}}(\bar{W}) & \frac{\partial f_1}{\partial z_1}(\bar{W}) & \cdots & \frac{\partial f_1}{\partial z_{\kappa+1}}(\bar{W}) \\ \frac{\partial f_2}{\partial w_1}(\bar{W}) & \frac{\partial f_2}{\partial w_2}(\bar{W}) & & & \cdots & & \frac{\partial f_2}{\partial z_{\kappa+1}}(\bar{W}) \\ \vdots & & & & \vdots & & \vdots \\ \frac{\partial f_{\kappa+1}}{\partial w_1}(\bar{W}) & & \cdots & \frac{\partial f_{\kappa+1}}{\partial w_{\kappa+1}}(\bar{W}) & \frac{\partial f_{\kappa+1}}{\partial z_1}(\bar{W}) & \cdots & \frac{\partial f_{\kappa+1}}{\partial z_{\kappa+1}}(\bar{W}) \\ \frac{\partial g_1}{\partial w_1}(\bar{W}) & \frac{\partial g_1}{\partial w_2}(\bar{W}) & \cdots & \frac{\partial g_1}{\partial w_{\kappa+1}}(\bar{W}) & \frac{\partial g_1}{\partial z_1}(\bar{W}) & \cdots & \frac{\partial g_1}{\partial z_{\kappa+1}}(\bar{W}) \\ \frac{\partial g_2}{\partial w_1}(\bar{W}) & & \cdots & & \frac{\partial g_2}{\partial z_1}(\bar{W}) & \cdots & \frac{\partial g_2}{\partial z_{\kappa+1}}(\bar{W}) \\ \vdots & & & & \vdots & & \vdots \\ \frac{\partial g_{\kappa+1}}{\partial w_1}(\bar{W}) & & \cdots & & \cdots & & \frac{\partial g_{\kappa+1}}{\partial z_{\kappa+1}}(\bar{W}) \end{pmatrix}.$$

$$\Gamma_{(2\kappa+2)\times(2\kappa+2)} = \begin{pmatrix} 0 & 0 & \cdots & & 0 & \frac{d}{\bar z^2} & \cdots & -\frac{2d}{\bar z^2} \\ 1 & 0 & & & \cdots & & & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & & \cdots & 1 & 0 & & \cdots & 0 \\ \frac{c}{\bar w^2} & 0 & \cdots & & -\frac{2c}{\bar w^2} & & \cdots & 0 \\ 0 & & \cdots & & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & & & & \cdots & & 1 & 0 \end{pmatrix}.$$

Let λ_i, for $i = 1, 2, \ldots, 2\kappa + 2$, denote the eigenvalues of the matrix Γ. Consider $J = \operatorname{diag}(l_1, l_2, \ldots, l_{2\kappa+2})$ as a diagonal matrix such that:

$$l_1 = l_{\kappa+2} = 1,$$

$$l_{1+i} = l_{\kappa+2+i} = 1 - i\epsilon, \quad 1 \le i \le \kappa,$$

$$0 < \epsilon < min\{\frac{1}{\kappa} - \frac{2d}{\kappa(\bar z^2 - d)}, \frac{1}{\kappa} - \frac{2c}{\kappa(\bar w^2 - c)}\}.$$

It is clear that the matrix J is nonsingular. By evaluating the product $J\Gamma J^{-1}$, we obtain

$$J\Gamma J^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{d}{\bar z^2} l_1 l_{\kappa+2}^{-1} & 0 & \cdots & -\frac{2d}{\bar z^2} l_1 l_{2\kappa+2}^{-1} \\ l_2 l_1^{-1} & 0 & & & \cdots & & & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & & \cdots & l_{\kappa+1} l_\kappa^{-1} & 0 & & \cdots & 0 \\ \frac{c}{\bar y^2} l_{\kappa+2} l_1^{-1} & 0 & \cdots & & -\frac{2c}{\bar y^2} l_{\kappa+2} l_{\kappa+1}^{-1} & & \cdots & 0 \\ 0 & & \cdots & & 0 & l_{\kappa+3} l_{\kappa+2}^{-1} & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & & & & \cdots & & l_{2\kappa+2} l_{2\kappa+1}^{-1} & 0 \end{pmatrix}.$$

We obtain:

$$0 < l_{\kappa+1} < l_\kappa < \cdots < l_2, \quad 0 < l_{2\kappa+2} < l_{2\kappa+1} < \cdots < l_{\kappa+3},$$

it implies,

$$l_2 l_1^{-1} < 1, \quad l_3 l_2^{-1} < 1, \quad \cdots \quad , \quad l_{\kappa+1} l_\kappa^{-1} < 1,$$

$$l_{\kappa+3} l_{\kappa+2}^{-1} < 1, \quad l_{\kappa+4} l_{\kappa+3}^{-1} < 1, \quad \cdots \quad , \quad l_{2\kappa+2} l_{2\kappa+1}^{-1} < 1.$$

Furthermore, we have,

$$\frac{d}{\bar{z}^2} l_1 l_{\kappa+2}^{-1} + \frac{2d}{\bar{z}^2} l_1 l_{2\kappa+2}^{-1} = \frac{d}{\bar{z}^2} + \frac{2d}{\bar{z}^2(1-\kappa\epsilon)} < 1,$$

and

$$\frac{c}{\bar{w}^2} l_{\kappa+2} l_1^{-1} + \frac{2c}{\bar{w}^2} l_{\kappa+2} l_{\kappa+1}^{-1} = \frac{c}{\bar{w}^2} + \frac{2c}{\bar{w}^2(1-\kappa\epsilon)} < 1.$$

It is straightforward to see that the matrix Γ shares the same eigenvalues as $J\Gamma J^{-1}$, and

$$\begin{aligned}\max_{1<i<2\kappa+1} |\lambda_i| \le |J\Gamma J^{-1}|_\infty &= \max_{1<i<2\kappa+1} \sum_{i=1}^{2\kappa+2} |\, a_{i,j} \,| \\ &= max\{l_2 l_1^{-1}, \cdots, l_{\kappa+2} l_{\kappa+1}^{-1}, \cdots, l_{2\kappa+2} l_{2\kappa+1}^{-1}, \frac{d}{\bar{z}^2} l_1 l_{\kappa+2}^{-1} + \frac{2d}{\bar{z}^2} l_1 l_{2\kappa+2}^{-1}, \\ &\frac{c}{\bar{w}^2} l_{\kappa+2} l_1^{-1} + \frac{2c}{\bar{w}^2} l_{\kappa+2} l_{\kappa+1}^{-1}\} < 1.\end{aligned}$$

Hence, the equilibrium point $(\bar{w}, \bar{z})$ of system (2.28) is locally asymptotically stable. ■

Lemma 2.4.5 *Consider equation* (2.28)*. If condition $pq > 3\sqrt{dc}$ holds, then the positive solution (w_n, z_n) of system* (2.28) *converges to the equilibrium point $(\bar{w}, \bar{z})$.*

Proof.

Starting from system (2.28), one can determine the corresponding positive equilibrium point.

$$(\bar{w}, \bar{z}) = \left(\frac{pq - c + d + \sqrt{(pq - d + c)^2 + 4pqd}}{2q}, \frac{pq - d + c + \sqrt{(pq - d + c)^2 + 4pqd}}{2p} \right).$$

Assume that (w_n, z_n) represents any positive solution of system (2.28). By taking into account relations (2.46)–(2.28), we obtain

$$\lim_{n\to\infty}\sup w_n = S_1, \lim_{n\to\infty}\inf w_n = s_1, \lim_{n\to\infty}\sup z_n = S_2, \quad \lim_{n\to\infty}\inf z_n = s_2. \tag{2.47}$$

where $s_i, S_i \in]0, +\infty[$ for $i = 1, 2$. Then,

$$\begin{aligned} S_1 - s_1 &= d\left(\frac{S_2}{s_2^2} - \frac{s_2}{S_2^2}\right) \\ &= d\frac{(S_2 - s_2)(S_2^2 + s_2S_2 + s_2^2)}{s_2^2S_2^2} \end{aligned}$$

Since $S_2 \geq p$, we obtain

$$S_1 - s_1 \leq \frac{3d(S_2 - s_2)S_2^2}{s_2^2S_2^2} = \frac{3d(S_2 - s_2)}{s_2^2}$$

Hence

$$S_1 - s_1 \leq \frac{3d}{p^2}(S_2 - s_2) \tag{2.48}$$

In the same way,

$$S_2 - s_2 \leq \frac{3c}{q^2}(S_1 - s_1) \tag{2.49}$$

From (2.48) and (2.49),

$$S_1 - s_1 \leq \frac{9dc}{p^2q^2}(S_1 - s_1) \tag{3}$$

Thus,

$$\left(1 - \frac{3d}{p^2} \cdot \frac{3c}{q^2}\right)(S_1 - s_1) \leq 0 \tag{2.50}$$

From the condition

$$pq > 3\sqrt{dc}$$

we have

$$p^2q^2 > 9dc$$

Then

$$1 > \frac{3d}{p^2} \cdot \frac{3c}{q^2}$$

Therefore,

$$1 - \frac{3d}{p^2} \cdot \frac{3c}{q^2} > 0$$

Then from (2.50),

$$S_1 - s_1 \leq 0$$

which gives

$$S_1 = s_1$$

From (2.49),

$$S_2 - s_2 \leq 0$$

thus

$$S_2 = s_2$$

Hence uniqueness is established.

This completes the proof. ■

By combining Lemma (2.4.4) with Lemma (2.4.5), we arrive at the following conclusion.

Theorem 2.4.2 *Consider equation* (2.28). *If condition* (2.29) *and* $pq > 3\sqrt{dc}$ *hold, then the single positive equilibrium* $(\bar{w}, \bar{z})$ *is globally asymptotically stable.*

Lemma 2.4.6 *If*

$$\beta_{l,\theta}^2 > \gamma_{r,\theta} > 1, \beta_{r,\theta}^2 > \gamma_{l,\theta} > 1, \forall \theta \in]0,1], \tag{2.51}$$

and, for $n = 0, 1, 2, \ldots,$

$$\frac{R_{n,\theta} L_{n-\kappa,\theta}^2}{L_{n,\theta} R_{n-\kappa,\theta}^2} \leq \frac{\gamma_{l,\theta}}{\gamma_{r,\theta}}, \forall \theta \in]0,1], \tag{2.52}$$

then every Pos. Fuz. Sol. of the Fuz. Diff. Eq. (2.1) *approaches the positive fuzzy equilibrium* y *as* $n \to +\infty$.

Proof. Suppose that there exists a Pos. Fuz. Num. y that satisfies (2.44). Using relation (2.44), together with conditions (2.51) and (2.52), we obtain

$$L_\theta = \beta_{l,\theta} + \frac{\gamma_{r,\theta} R_\theta}{R_\theta^2}, \quad R_\theta = \beta_{r,\theta} + \frac{\gamma_{l,\theta} L_\theta}{L_\theta^2}, \tag{2.53}$$

hence we can from (2.53) have that

$$\begin{cases} L_\theta = \frac{\beta_{l,\theta}\beta_{r,\theta}+\gamma_{r,\theta}-\gamma_{l,\theta}+\sqrt{\left(\beta_{l,\theta}\beta_{r,\theta}+\gamma_{r,\theta}-\gamma_{l,\theta}\right)^2+4\beta_{l,\theta}\beta_{r,\theta}\gamma_{l,\theta}}}{2\beta_{r,\theta}}, \\ R_\theta = \frac{\beta_{l,\theta}\beta_{r,\theta}+\gamma_{l,\theta}-\gamma_{r,\theta}+\sqrt{\left(\beta_{r,\theta}\beta_{r,\theta}+\gamma_{l,\theta}-\gamma_{r,\theta}\right)^2+4\beta_{l,\theta}\beta_{r,\theta}\gamma_{r,\theta}}}{2\beta_{l,\theta}}. \end{cases} \tag{2.54}$$

Assume that y_n represents a Pos. Fuz. Sol. of the Fuz. Diff. Eq. (2.1) that fulfills condition (2.2). By taking into account relations (2.52) and (2.53), and applying Lemma (2.4.4) together with Lemma (2.4.5), we deduce that

$$\lim_{n\to\infty} L_{n,\theta} = L_\theta, \quad \lim_{n\to\infty} R_{n,\theta} = R_\theta,$$

then

$$\lim_{n\to\infty} D\left(y_n, y\right) = \lim_{n\to\infty} \sup_{\theta\in]0,1]} \left\{\max\left\{| L_{n,\theta} - L_\theta |, | R_{n,\theta} - R_\theta |\right\}\right\} = 0.$$

■

2.5 Rate of convergence

Now, we turn our attention to analyzing the rate of convergence associated with the fuzzy differential equation (2.1). This step is crucial for understanding how efficiently the proposed numerical or analytical method approaches the exact fuzzy solution as the number of iterations or discretization steps increases. By examining the convergence rate, we can assess the accuracy, stability, and overall effectiveness of the approach in approximating the true behavior of the fuzzy system described by equation (2.1).

If Case (i) holds, the following statement is valid:

Lemma 2.5.1 *For any solution $\{u_n\}$ of equation (2.15), the following asymptotic relations are satisfied:*

$$\lim_{n\to\infty} \left| \frac{u_{n+1} - \bar{u}}{u_n - \bar{u}} \right| = |\lambda_j|,$$
$$\lim_{n\to\infty} \sup \left(|u_n - \bar{u}|\right)^{1/n} = |\lambda_j|,$$

where $j \in \{1, 2, \ldots, \kappa\}$ and λ_j denote the roots of the characteristic polynomial associated with (2.39).

Proof.

We get from equation (2.15):

$$\begin{aligned} u_{n+1} - \bar{u} &= \left(d + c\frac{u_n}{u_{n-\kappa}^2}\right) - \left(d + c\frac{\bar{u}}{\bar{u}^2}\right) \\ &= \frac{c}{u_{n-\kappa}^2}\left(u_n - \bar{u}\right) - \frac{c\left(\bar{u} + u_{n-\kappa}\right)}{\bar{u} \cdot u_{n-\kappa}^2}\left(u_{n-\kappa} - \bar{u}\right). \end{aligned}$$

Set $e_n = u_n - \bar{u}$. Then the sequence $\{e_n\}$ satisfies

$$e_{n+1} + p_n e_n + q_n e_{n-\kappa} = 0,$$

where

$$p_n = -\frac{c}{z_{n-\kappa}^2}, \qquad q_n = \frac{c(\bar{u} + u_{n-\kappa})}{\bar{u} \cdot u_{n-\kappa}^2}.$$

Since the equilibrium $\bar{u}$ of equation (2.15) is globally asymptotically stable, it follows that

$$\lim_{n\to\infty} p_n = -\frac{c}{\bar{u}^2}, \qquad \lim_{n\to\infty} q_n = \frac{2c}{\bar{u}^2}.$$

Therefore, the limiting form of equation (2.15) coincides with the linearized equation (2.37). ■

Theorem 2.5.1 *Consider the Fuz. Diff. Eq.* (2.1)*, where* $\beta, \gamma \in \mathbb{R}_{\mathcal{F}}^+$*,* $y_{-\kappa}, y_{-\kappa+1}, \ldots, y_0 \in \mathbb{R}_{\mathcal{F}}^+$*, and* $\kappa \in \{2, 3, \ldots\}$*. Then every solution* $\{y_n\}$ *satisfies the following asymptotic relations:*

$$\lim_{n\to\infty} \frac{\| y_{n+1} - y \|}{\| y_n - y \|} = \mid \lambda_j \mid \tag{2.55}$$

$$\lim_{n\to\infty} \left(\| y_n - y \|\right)^{1/n} = \mid \lambda_j \mid \tag{2.56}$$

where $j \in \{1, \ldots, \kappa\}$ *and* λ_j *denote the roots of the characteristic polynomial associated with the linearized form of the Fuz. Diff. Eq.* (2.1).

Proof. Assume that there exist Fuz. Nums. y_n and y such that:

$$y_n = \beta + \frac{\gamma y_n}{y_{n-\kappa}^2}, \quad [y_n]_\theta = [L_{n,\theta}, R_{n,\theta}], \quad \theta \in]0, 1]. \tag{2.57}$$

$$y = \beta + \frac{\gamma y}{y^2}, \quad [y]_\theta = [L_\theta, R_\theta], \quad \theta \in]0,1]. \tag{2.58}$$

In which $L_\theta, R_\theta \geq 0$. One gets

$$L_{n+1,\theta} = \beta_{l,\theta} + \frac{\gamma_{l,\theta} L_{n,\theta}}{L^2_{n-\kappa,\theta}}, \quad R_{n+1,\theta} = \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_{n,\theta}}{R^2_{n-\kappa,\theta}} \tag{2.59}$$

$$L_\theta = \beta_{l,\theta} + \frac{\gamma_{l,\theta} L_\theta}{L^2_\theta}, \quad R_\theta = \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_\theta}{R^2_\theta}. \tag{2.60}$$

Therefore, from Lemma (2.5.1), we have:

$$\lim_{n\to\infty} \mid \frac{L_{n+1} - L_\theta}{L_n - L_\theta} \mid = \mid \lambda_{j,l,\theta} \mid, \quad \lim_{n\to\infty} \mid \frac{R_{n+1} - R_\theta}{R_n - R_\theta} \mid = \mid \lambda_{j,r,\theta} \mid \tag{2.61}$$

$$\lim_{n\to\infty} \sup \left(\mid L_n - L_\theta \mid\right)^{1/n} = \mid \lambda_{j,l,\theta} \mid, \quad \lim_{n\to\infty} \sup \left(\mid R_n - R_\theta \mid\right)^{1/n} = \mid \lambda_{j,r,\theta} \mid \tag{2.62}$$

Such that $\lambda_j \in \mathbb{R}^+_{\mathcal{F}}$ and $[\lambda_j]_\theta = \left[\lambda_{j,l,\theta}, \lambda_{j,r,\theta}\right]$.

The analysis for the alternative case proceeds in a similar manner, and hence its proof is omitted. ■

2.6 Numerical examples

In this section, we present several illustrative examples to demonstrate the applicability and validity of the theoretical results obtained in the previous sections. These examples serve to highlight how the proposed approach can be effectively employed to solve practical problems involving fuzzy differential equations. By applying the developed methods to specific test cases, we aim to verify the accuracy of the theoretical analysis and to illustrate the consistency between the analytical findings and numerical outcomes. Moreover, these examples help clarify the implementation process and showcase the robustness of the proposed framework under different parameter settings.

Example 2.6.1 *Let us consider the Fuz. Diff. Eq. (2.1), where β, γ, and the initial conditions*

y_i, $i \in \{-3,-2,-1,0\}$, are assumed to be Pos. Fuz. Nums.. In this case, we take $\kappa = 3$, and thus obtain:

$$\beta(x) = \begin{cases} x-4, & 4 \le x \le 5 \\ -x+6, & 5 \le x \le 6 \end{cases}, \gamma(x) = \begin{cases} x-5, & 5 \le x \le 6 \\ -x+7, & 6 \le x \le 7 \end{cases}$$

$$y_{-3}(x) = \begin{cases} 3x-5, & \frac{5}{3} \le x \le 2 \\ -3x+7, & 2 \le x \le \frac{7}{3} \end{cases}, y_{-2}(x) = \begin{cases} \frac{1}{2}x - \frac{3}{2}, & 3 \le x \le 5 \\ -\frac{1}{4}x - \frac{9}{4}, & 5 \le x \le 9 \end{cases}$$

$$y_{-1}(x) = \begin{cases} \frac{1}{3}x - \frac{1}{3}, & 1 \le x \le 4 \\ -\frac{1}{3}x + \frac{7}{3}, & 4 \le x \le 7 \end{cases}, y_0(x) = \begin{cases} x-2, & 2 \le x \le 3 \\ -x+4, & 3 \le x \le 4 \end{cases}$$

We have

$$[\beta]_\theta = [\theta+4, 6-\theta], \quad [\gamma]_\theta = [\theta+5, 7-\theta],$$

$$[y_{-3}]_\theta = \left[\frac{1}{3}\theta + \frac{5}{3}, \frac{7}{3} - \frac{1}{3}\theta\right], \quad [y_{-2}]_\theta = [3+2\theta, 9-4\theta], \quad \theta \in]0,1],$$

$$[y_{-1}]_\theta = [1+3\theta, 7-3\theta], \quad [y_0]_\theta = [2+\theta, 4-\theta],$$

therfore, it follows that

$$\begin{cases} \overline{\bigcup_{\theta\in]0,1]} [\beta]_\theta} = [4,6], \quad \overline{\bigcup_{\theta\in]0,1]} [\gamma]_\theta} = [5,7], \\ \\ \overline{\bigcup_{\theta\in]0,1]} [y_{-3}]_\theta} = [\frac{5}{3}, \frac{7}{3}], \quad \overline{\bigcup_{\theta\in]0,1]} [y_{-2}]_\theta} = [3,9], \\ \\ \overline{\bigcup_{\theta\in]0,1]} [y_{-1}]_\theta} = [1,7], \quad \overline{\bigcup_{\theta\in]0,1]} [y_0]_\theta} = [2,4]. \end{cases} \tag{2.63}$$

It is straightforward to verify that Case (i) applies. Consequently, equation (2.1) can be expressed as a system of difference equations depending on the parameter θ,

$$L_{n+1,\theta} = \beta_{l,\theta} + \frac{\gamma_{l,\theta} L_{n,\theta}}{L^2_{n-3,\theta}}, \quad R_{n+1,\theta} = \beta_{r,\theta} + \frac{\gamma_{r,\theta} R_{n,\theta}}{R^2_{n-3,\theta}}, \quad \theta \in]0,1]. \tag{2.64}$$

Since the conditions $\beta^2_{l,\theta} > \frac{4}{3}\gamma_{l,\theta}$ and $\beta^2_{r,\theta} > \frac{4}{3}\gamma_{r,\theta}$ hold for all $\theta \in]0,1]$, and given that the *initial values $y_i \in \mathbb{R}_{\mathcal{F}}^{+}$ for $i = -3, -2, -1, 0$, Theorem (2.3.1) ensures that every positive solution y_n of equation (2.1) remains bounded and persistent.*

Furthermore, by Theorem (2.4.1), the system admits a unique positive equilibrium, and each Pos. Fuz. Sol. y_n converges to $\bar{y}$ as $n \to \infty$. Figure 2.1 illustrates the time evolution of the Fuz. Sol.

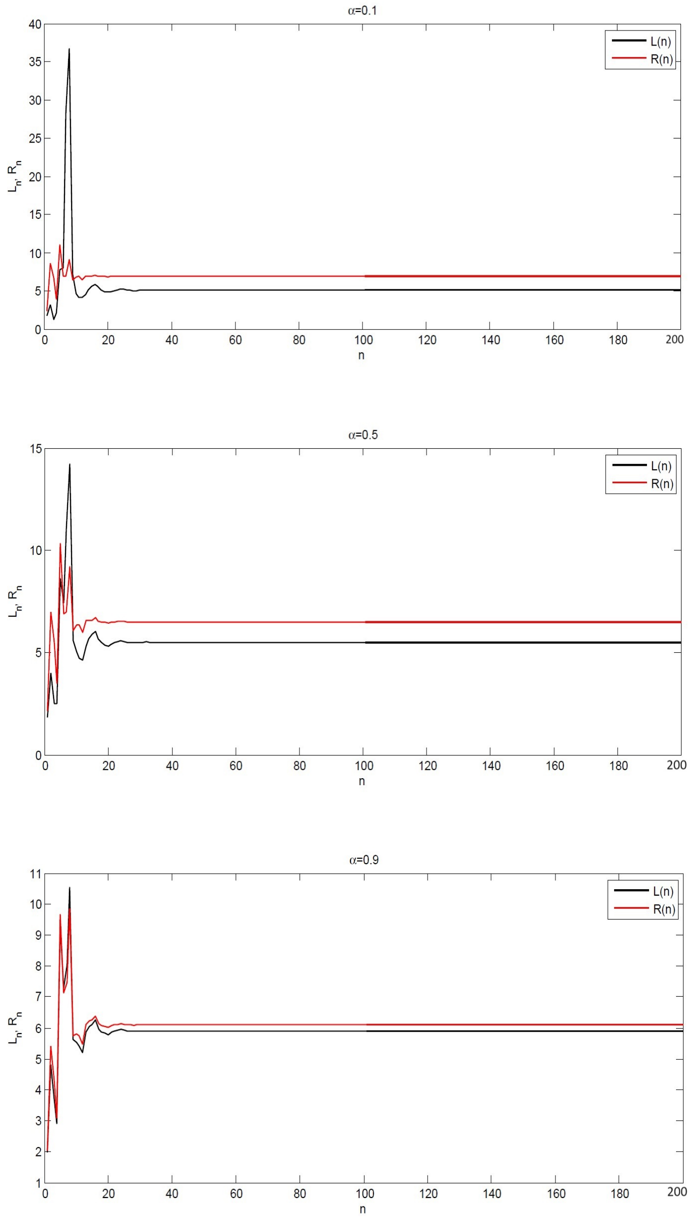


Figure 2.1: The solution of the system (2.1) at $\theta = 0.1$, $\theta = 0.5$ and $\theta = 0.9$

Example 2.6.2 *In this setting, we take $\kappa = 4$. Referring to equation (2.1), the parameters β, γ, as well as the initial values y_i for $i \in \{-4,-3,-2,-1,0\}$, are assumed to be Pos. Fuz. Nums. Thus, we obtain:*

$$\beta(x) = \begin{cases} x-5, & 5 \le x \le 6 \\ -2x+14, & 6 \le x \le 7 \end{cases}, \quad \gamma(x) = \begin{cases} x-4, & 4 \le x \le 4.5 \\ -x+6, & 4.5 \le x \le 6 \end{cases}$$

$$y_{-4}(x) = \begin{cases} 2x-4, & 2 \le x \le 2.5 \\ -2x+6, & 2.5 \le x \le 3 \end{cases}, \quad y_{-3}(x) = \begin{cases} x-4, & 4 \le x \le 5 \\ -x+6, & 5 \le x \le 6 \end{cases}.$$

$$y_{-2}(x) = \begin{cases} \frac{1}{5}x - \frac{6}{5}, & 6 \le x \le 11 \\ -\frac{1}{2}x + \frac{13}{2}, & 11 \le x \le 13 \end{cases}, \quad y_{-1}(x) = \begin{cases} x-2, & 2 \le x \le 3 \\ -\frac{1}{5}x + \frac{8}{5}, & 3 \le x \le 8 \end{cases}$$

$$y_0(x) = \begin{cases} x-1, & 1 \le x \le 2 \\ -x+3, & 2 \le x \le 3 \end{cases}$$

We have for all $\theta \in]0,1]$

$$[\beta]_\theta = [\theta + 5, 7 - \frac{1}{2}\theta], \quad [\gamma]_\theta = [4+\theta, 6-\theta],$$

$$[y_{-4}]_\theta = \left[2 + \frac{1}{2}\theta, 3 - \frac{1}{2}\theta\right], \quad [y_{-3}]_\theta = [4+\theta, 6-\theta],$$

$$[y_{-2}]_\theta = [6+5\theta, 13-2\theta], \quad [y_{-1}]_\theta = [2+\theta, 8-5\theta],$$

$$[y_0]_\theta = [1+\theta, 3-\theta].$$

Therfore, it follows that

$$\begin{cases} \overline{\bigcup_{\theta\in]0,1]} [\beta]_\theta} = [5,7], \quad \overline{\bigcup_{\theta\in]0,1]} [\gamma]_\theta} = [4,6], \\ \\ \overline{\bigcup_{\theta\in]0,1]} [y_{-4}]_\theta} = [2,3], \quad \overline{\bigcup_{\theta\in]0,1]} [y_{-3}]_\theta} = [4,6], \\ \\ \overline{\bigcup_{\theta\in]0,1]} [y_{-2}]_\theta} = [6,13], \quad \overline{\bigcup_{\theta\in]0,1]} [y_{-1}]_\theta} = [2,8], \\ \\ \overline{\bigcup_{\theta\in]0,1]} [y_0]_\theta} = [1,3]. \end{cases} \tag{2.65}$$

As Case (ii) clearly holds, equation (2.1) gives rise to a system of difference equations where θ serves as a parameter:

$$L_{n+1,\theta} = \beta_{l,\theta} + \frac{\gamma_{r,\theta} R_{n,\theta}}{R^2_{n-4,\theta}}, \quad R_{n+1,\theta} = \beta_{r,\theta} + \frac{\gamma_{l,\theta} L_{n,\theta}}{L^2_{n-4,\theta}}, \quad \theta \in (0,1]. \tag{2.66}$$

Furthermore, if the conditions $\beta^2_{l,\theta} > \frac{4}{3}\gamma_{l,\theta}$ and $\beta^2_{r,\theta} > \frac{4}{3}\gamma_{r,\theta}$ hold for all $\theta \in (0,1]$, together with $y_i \in \mathbb{R}^+_{\mathcal{F}}$ for $i \in \{-4,-3,-2,-1,0\}$, then by applying Theorem (2.3.1), it follows that every positive solution y_n of equation (2.1) remains bounded and persistent.

In addition, Theorem (2.4.1) guarantees the existence of a unique positive equilibrium. Moreover, every Pos. Fuz. Sol. y_n of equation (2.1) converges to $\bar{y}$ as $n \to \infty$.

Analogous to Figure 2.1, Figure 2.2 illustrates the numerical validation of these theoretical results by displaying the dynamics of the Fuz. Sol. under a distinct set of initial conditions. This confirms both the stability and convergence behavior of the system.

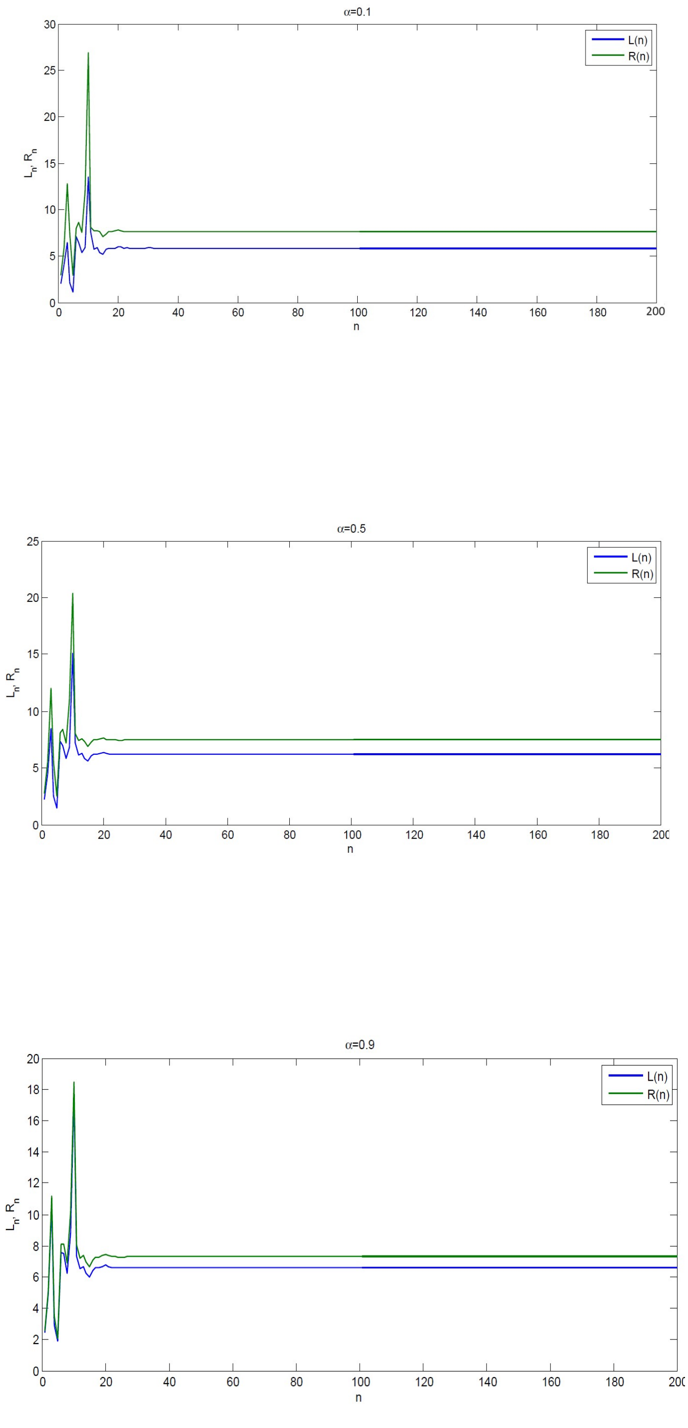


Figure 2.2: The solution of the system (2.1) at $\theta = 0.1$, $\theta = 0.5$ and $\theta = 0.9$

2.7 Conclusion

This chapter is devoted to a comprehensive qualitative study of a higher-order fuzzy difference equation with a quadratic term, given by

$$y_{n+1} = \beta + \frac{\gamma\, y_n}{y_{n-\kappa}^2}, \qquad n = 0, 1, \ldots, \qquad \kappa \geq 2,$$

where the parameters β, γ, and the initial values $y_{-\kappa}, \ldots, y_0$ are assumed to be positive fuzzy numbers. The equation represents a natural extension of classical rational difference equations, but with an additional layer of uncertainty that is captured using the fuzzy framework. The chapter establishes several fundamental properties of the fuzzy solutions and examines their global behavior through θ-level representation, interval analysis, and the use of 1-division of fuzzy numbers.

The chapter begins by proving the existence and uniqueness of a positive fuzzy solution to equation (2.1). This is achieved through converting the equation into its θ-cut form and applying the arithmetic of fuzzy intervals together with the notion of 1-division. It is shown that for any positive fuzzy initial data, the recurrence generates a well-defined sequence of fuzzy numbers. This is a crucial first step, since the well-posedness of the recursive model ensures that all subsequent qualitative results are meaningful.

The boundedness and persistence of solutions are then investigated under two main sets of conditions on β and γ. By analyzing the lower and upper θ-cut sequences, the chapter provides sufficient conditions under which all fuzzy solutions remain positive and confined to invariant fuzzy intervals. The results confirm that despite the nonlinearity and the presence of a quadratic term in the denominator, the solutions avoid both extinction and blow-up. This analysis demonstrates how the fuzzy structure interacts with the nonlinear dynamics of the model, leading to behaviors that closely parallel the crisp case while preserving the uncertainty inherent in the system.

A significant part of the chapter is devoted to the study of convergence toward the unique equilibrium point $\bar{y}$. By transforming the fuzzy recurrence into a system of two crisp interval recurrences at each θ level, the chapter employs classical stability techniques to establish the asymptotic convergence of all positive fuzzy solutions. The condition $\beta^2 > \frac{4}{3}\gamma$ emerges as a key threshold that separates stable from unstable dynamics. The analysis shows that convergence is not only global but also monotone in many parameter settings, and that the structure of the θ-cuts preserves convergence across all levels of fuzziness.

The chapter further explores the rate of convergence. By examining the difference between y_n and the equilibrium $\bar{y}$ in the θ-cut setting, it is shown that convergence can be geometric under additional parameter restrictions. These results highlight that the behavior of the fuzzy system mirrors its crisp counterpart in many ways, but with additional richness arising from the interaction between the width of the fuzzy intervals and the nonlinear nature of the equation.

To illustrate the theoretical results, several numerical examples are presented at the end of the chapter. These examples use different fuzzy initial values and parameter settings to demonstrate boundedness, convergence, and the effect of fuzziness on the long-term behavior of the solutions. The numerical simulations confirm the theoretical predictions and reveal how the θ-cuts contract over time as the solutions approach the equilibrium point.

Overall, Chapter 2 forms a cornerstone of the dissertation. It provides a unified methodological and theoretical framework for analyzing higher-order fuzzy difference equations. The results obtained here serve as a foundation for Chapter 3, where a more general class of power-type fuzzy equations is studied, and for Chapter 4, which extends the analysis to multidimensional nonlinear rational systems. Thus, the chapter not only stands as an independent contribution but also plays a key structural role in linking the preliminary theory of Chapter 1 with the advanced results of the subsequent chapters.

Chapter 3

Analysis of the global properties of nonlinear higher-order fuzzy difference equations

3.1 Introduction

The study of Fuz. Diff. Eqs. was first introduced by Kandel and Byatt [41, 42], while Kaleva [43, 44] offered a detailed exposition on Fuz. Diff. Eqs. and their connection with initial value (Cauchy) problems. Later, Zhang et al. [79] examined the conditions under which positive solutions exist for Fuz. Diff. Eqs. and showed that such solutions remain bounded and persistent.

Applications of Fuz. Diff. Eqs. have also appeared in finance. For instance, Chrysafis et al. [11] proposed their use as an alternative framework for analyzing the time value of money.

Since mathematical models often involve incomplete or imprecise information, fuzzy set theory has proven to be an efficient approach to capture uncertainty and vagueness within such models. Consequently, there has been growing attention toward understanding the dynamics of Fuz. Diff. Eq. systems, where both parameters and initial data are expressed as Fuz. Nums., leading to solution sequences composed of Fuz. Nums. (see, e.g., [5, 12, 25, 37, 58]).

In [70], Yalçinkaya et al. investigated the global behavior of Fuz. Diff. Eqs..

Building on these contributions, this work focuses on a generalized form of Fuz. Diff. Eqs. and aims to analyze the qualitative properties of Pos. Fuz. Sols. to the following higher-order Fuz. Diff. Eq.:

$$\omega_{n+1} = \frac{A\omega_{n-k}}{1+\omega_{n-1}^{p}}, \quad n \in \mathbb{N}_0,\ k \in \mathbb{N}_2, \tag{3.1}$$

where A and the initial values $\omega_{-k}, \omega_{-k+1}, \ldots, \omega_0$ are assumed to be Pos. Fuz. Nums..

3.2 Existence of the unique Pos. Fuz. Sol.

Theorem 3.2.1 *Let us consider the Fuz. Diff. Eq. (3.1), where $A \in \mathbb{R}_{\mathcal{F}}^{+}$, and the initial conditions $\omega{-}k, \omega_{-k+1}, \ldots, \omega_0$ belong to $\mathbb{R}_{\mathcal{F}}^{+}$ with $k \in 2, 3, \ldots$. Then, the equation (3.1) admits a **unique Pos. Fuz. Sol.**.*

Proof.

1. **Existence of the solution:** Consider the θ-cuts with $\theta \in \,]0,1]$ and $n \in \mathbb{N}$. We define

$$\begin{cases} [\omega_n]_\theta = [L_{n,\theta}, R_{n,\theta}]\,, \\ [A]_\theta = [A_{l,\theta}, A_{r,\theta}]\,. \end{cases} \tag{3.2}$$

By applying Lemma (1.4.1), together with equations (3.1) and (**??**), we can conclude that

$$\begin{aligned}
[\omega_{n+1}]_\theta &= [L_{n+1,\theta}, R_{n+1,\theta}] \\
&= \left[\frac{A\omega_{n-k}}{1+\omega_{n-1}^p}\right]_\theta \\
&= \frac{[A]_\theta \times [\omega_{n-k}]_\theta}{\left[1+\omega_{n-1}^p\right]_\theta} \\
&= \frac{[A_{l,\theta}L_{n-k,\theta}, A_{r,\theta}R_{n-k,\theta}]}{\left[1+L_{n-1,\theta}^p, 1+R_{n-1,\theta}^p\right]} \\
&= \left[\frac{A_{l,\theta}L_{n-k,\theta}}{1+R_{n-1,\theta}^p}, \frac{A_{r,\theta}R_{n-k,\theta}}{1+L_{n-1,\theta}^p}\right],
\end{aligned}$$

in which we obtain

$$L_{n+1,\theta} = \frac{A_{l,\theta}L_{n-k,\theta}}{1+R_{n-1,\theta}^p}, \quad R_{n+1,\theta} = \frac{A_{r,\theta}R_{n-k,\theta}}{1+L_{n-1,\theta}^p}. \tag{3.3}$$

It follows that for any prescribed initial data $(L_{j,\theta}, R_{j,\theta})$, $j = -k, -k+1, \ldots, 0$, with $\theta \in]0,1]$, there exists a unique pair $(L_{n,\theta}, R_{n,\theta})$ that provides the solution.

$$[\omega_n]_\theta = [L_{n,\theta}, R_{n,\theta}]\,. \tag{3.4}$$

Because x_j, $j = -k, -k+1, \ldots, 0$, are Pos. Fuz. Nums., then for any $\theta_i \in]0,1]$, $i = 1,2$, with $\theta_1 \leq \theta_2$, one obtains the inequalities:

$$0 < L_{j,\theta_1} \leq L_{j,\theta_2} \leq R_{j,\theta_2} \leq R_{j,\theta_1}\ 0 < A_{l,\theta_1} \leq A_{l,\theta_2} \leq A_{r,\theta_2} \leq A_{r,\theta_1}. \tag{3.5}$$

We establish by induction for $n \leq k$, that:

$$0 < L_{n,\theta_1} \leq L_{n,\theta_2} \leq R_{n,\theta_2} \leq R_{n,\theta_1} \tag{3.6}$$

we obtain

$$0 < L_{m-k,\theta_1} \leq L_{m-k,\theta_2} \leq R_{m-k,\theta_2} \leq R_{m-k,\theta_1}.$$

This further yields

$$1+L^p_{m-k,\theta_1} \leq 1+L^p_{m-k,\theta_2} \leq 1+R^p_{m-k,\theta_2} \leq 1+R^p_{m-k,\theta_1},$$

and consequently

$$\frac{1}{1+R^p_{m-k,\theta_1}} \leq \frac{1}{1+R^p_{m-k,\theta_2}} \leq \frac{1}{1+L^p_{m-k,\theta_2}} \leq \frac{1}{1+L^p_{m-k,\theta_1}}.$$

It follows that

$$L_{m+1,\theta_1} = \frac{A_{l,\theta}\, L_{m-k,\theta_1}}{1+R^p_{m-1,\theta_1}} \leq \frac{A_{l,\theta_2}\, L_{m-k,\theta_2}}{1+R^p_{m-1,\theta_2}} = L_{m+1,\theta_2}$$

and

$$L_{m+1,\theta_2} \leq \frac{A_{r,\theta_2}\, R_{m-k,\theta_2}}{1+L^p_{m-1,\theta_2}} = R_{m+1,\theta_2} \leq \frac{A_{r,\theta_1}\, R_{m-k,\theta_1}}{1+L^p_{m-1,\theta_1}} = R_{m+1,\theta_1}.$$

Thus,

$$L_{m+1,\theta_1} \leq L_{m+1,\theta_2} \leq R_{m+1,\theta_2} \leq R_{m+1,\theta_1}. \tag{3.7}$$

Hence, (3.6) holds. Furthermore, from (3.3), we have

$$L_{1,\theta} = \frac{A_{l,\theta}L_{-k,\theta}}{1+R^p_{-1,\theta}}, \quad R_{1,\theta} = \frac{A_{r,\theta}R_{-k,\theta}}{1+L^p_{-1,\theta}} \tag{3.8}$$

- According to Lemma (1.4.2), if $x_j \in \mathbb{R}^+_{\mathcal{F}}$ for $j=-k,-k+1,\ldots,0$, then for any $A \in \mathbb{R}^+_{\mathcal{F}}$, the functions $L_{j,\theta}$ and $R_{j,\theta}$ are left-continuous. Consequently, from (3.3), both $L_{1,\theta}$ and $R_{1,\theta}$ are also left-continuous. By induction, this property extends to $L_{n,\theta}$ and $R_{n,\theta}$ for all $n \geq 1$.
- From (3.7), one deduces that L_θ is a non-decreasing function, while R_θ is non-increasing.
- Lemma (1.4.2) further ensures that $L_{n,\theta}$ and $R_{n,\theta}$ are right-continuous at 0.
- Finally, combining (3.7) with (3.8), it follows that $L_1 \leq R_1$.

Since the assumptions of Lemma (1.4.2) are satisfied, it follows that there exists a

Fuz. Num. $x_n \in \mathbb{R}^+_{\mathcal{F}}$ such that its θ-cut is given by

$$[x_n]_\theta = [L_{n,\theta}, R_{n,\theta}].$$

2. Positivity of the Solution

To verify that the solution represents a Pos. Fuz. Num., it is sufficient to prove that the support of ω_n, denoted by

$$\text{supp } \omega_n = \overline{\bigcup_{\theta\in]0,1]} [L_{n,\theta}, R_{n,\theta}]},$$

is compact.

Since $\omega_j \in \mathbb{R}^+_{\mathcal{F}}$ for $j = -k, -k+1, \ldots, 0$ and $A \in \mathbb{R}^+_{\mathcal{F}}$, there exist constants $M_A, N_A > 0$ and $M_j, N_j > 0$ for $j = -k, -k+1, \ldots, 0$, such that for all $\theta \in]0,1]$ we have:

$$\begin{cases} [A_{l,\theta}, A_{r,\theta}] \subset [M_A, N_A], \\ \left[L_{j,\theta}, R_{j,\theta}\right] \subset \left[M_j, N_j\right]. \end{cases} \tag{3.9}$$

Consequently, from (3.8) and (3.9), it follows:

$$[L_{1,\theta}, R_{1,\theta}] \subset \left[\frac{M_A M_{-k}}{1+N^p_{-1}}, \frac{N_A N_{-k}}{1+M^p_{-1}}\right], \quad \theta \in]0,1]. \tag{3.10}$$

Thus, we obtain:

$$\bigcup_{\theta\in]0,1]} [L_{1,\theta}, R_{1,\theta}] \subset \left[\frac{M_A M_{-k}}{1+N^p_{-1}}, \frac{N_A N_{-k}}{1+M^p_{-1}}\right], \quad \theta \in]0,1]. \tag{3.11}$$

Consequently, $\overline{\bigcup_{\theta\in]0,1]} [L_{1,\theta}, R_{1,\theta}]}$ is compact, and $\overline{\bigcup_{\theta\in]0,1]} [L_{1,\theta}, R_{1,\theta}]} \subset]0,\infty[$. Through induction, it can be inferred that $\overline{\bigcup_{\theta\in]0,1]} [L_{n,\theta}, R_{n,\theta}]}$ is compact.
Additionally, for $n = 1, 2, \ldots$,

$$\overline{\bigcup_{\theta\in]0,1]} [L_{n,\theta}, R_{n,\theta}]} \subset]0,\infty[\tag{3.12}$$

From (3.12), we deduce that $[L_{n,\theta}, R_{n,\theta}]$ generates a positive sequence ω_n fulfilling condition (3.4).

3. Uniqueness of the solution

Assume that, under the initial conditions $x_j \in \mathbb{R}^+_{\mathcal{F}}$ for $j = -k, -k+1, \cdots, 0$, there exists another solution $\overline{\omega_n}$ of (3.1). By reasoning in a similar manner as before, one obtains, for all $n \in \mathbb{N}^*$,

$$[\overline{\omega_n}]_\theta = [L_{n,\theta}, R_{n,\theta}], \quad \theta \in]0,1]$$

It implies that $[\overline{\omega_n}]_\theta = [\omega_n]_\theta$, $\theta \in]0,1]$ so $\forall n \in \mathbb{N}^*$, $\omega_n = \overline{\omega_n}$.

■

3.3 Equilibrium points in different cases

In this section, we'll proceed to determine the equilibrium points of the system given by:

$$u_{n+1} = \frac{r u_{n-k}}{1 + v^p_{n-1}}, \quad v_{n+1} = \frac{s v_{n-k}}{1 + u^p_{n-1}}, \quad n = 0, 1, \ldots, \tag{3.13}$$

Theorem 3.3.1 *Let equation* (3.13) *be considered, then we have the following cases fo the equilibrium points:*

(i) The pair $(\overline{u}_0, \overline{v}_0) = (0,0)$ *is always an equilibrium point of system* (3.13).

(ii) When $r, s \in]1, +\infty[$, *the system* (3.13) *admits the equilibrium point*

$$(\overline{u}_1, \overline{v}_1) = \left((s-1)^{\frac{1}{p}}, (r-1)^{\frac{1}{p}}\right).$$

(iii) If $r \in]1, +\infty[$ *and* $s = 1$*, then the system* (3.13) *possesses the equilibrium point*

$$(\overline{u}_2, \overline{v}_2) = \left(0,\ (r-1)^{\frac{1}{p}}\right).$$

(iv) If $r = 1$ *and* $s \in]1, +\infty[$*, then the system* (3.13) *has the equilibrium point*

$$(\overline{u}_3, \overline{v}_3) = \left((s-1)^{\frac{1}{p}},\ 0\right).$$

Proof. The result follows directly from the definition of an equilibrium point.

(i) $\forall r, s \in]0, +\infty[$, we have

$$\begin{aligned}
&\bar{u} = \frac{r\,\bar{u}}{1+\bar{v}^p},\\
&\bar{u}(1+\bar{v}^p) - r\bar{u} = 0,\\
&\bar{u}\,(1+\bar{v}^p - r) = 0,\\
&\bar{u} = 0, \quad \text{or} \quad \bar{v}^p = r-1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\bar{v} = \frac{s\,\bar{v}}{1+\bar{u}^p},\\
&\bar{v} - s\bar{v} = 0,\\
&\bar{v}(1-s) = 0,\\
&\bar{v} = 0, \quad \text{or} \quad s = 1.
\end{aligned}$$

So, we have, $(\bar{u}, \bar{v}) = (0, 0)$ is always an equilibrieum point of (3.13)

(ii) If $r, s \in]1, +\infty[$

$$
\begin{aligned}
&\bar{u} = \frac{r\,\bar{u}}{1+\bar{v}^p},\\
&\bar{u}(1+\bar{v}^p) - r\bar{u} = 0,\\
&\bar{u}\,(1+\bar{v}^p - r) = 0,\\
&1+\bar{v}^p - r = 0,\\
&\bar{v} = (r-1)^{\frac{1}{p}},\\
&\bar{v} = \frac{s\,\bar{v}}{1+\bar{u}^p},\\
&(r-1)^{\frac{1}{p}} = \frac{s\,(r-1)^{\frac{1}{p}}}{1+\bar{u}^p},\\
&1+\bar{u}^p = s,\\
&\bar{u} = (s-1)^{\frac{1}{p}}.
\end{aligned}
$$

So, we have, $(\bar{u}, \bar{v}) = ((s-1)^{\frac{1}{p}}, (r-1)^{\frac{1}{p}})$

(iii) If $r \in]1, +\infty[$ and $s = 1$

$$
\begin{aligned}
&\bar{u} = \frac{r\bar{u}}{1+\bar{v}^p},\\
&\bar{u}(1+\bar{v}^p) - r\bar{u} = 0,\\
&\bar{u}(1+\bar{v}^p - r) = 0,\\
&1+\bar{v}^p - r = 0,
\end{aligned}
$$

so, we have

$$\bar{v} = (r-1)^{\frac{1}{p}},$$

thus

$$\begin{aligned}
\bar{v} &= \frac{\bar{v}}{1+\bar{u}^p},\\
(r-1)^{\frac{1}{p}} &= \frac{(r-1)^{\frac{1}{p}}}{1+\bar{u}^p},\\
(r-1)^{\frac{1}{p}}&(1+\bar{u}^p-1) = 0,
\end{aligned}$$

which gives

$$\bar{u} = 0.$$

So, we have $(\bar{u}, \bar{v}) = \left(0, (r-1)^{\frac{1}{p}}\right)$.

(iv) If $s \in]1, +\infty[$ and $r = 1$

$$\begin{aligned}
\bar{v} &= \frac{s\bar{v}}{1+\bar{u}^p},\\
\bar{v}&(1+\bar{u}^p) - s\bar{v} = 0,\\
\bar{v}&(1+\bar{u}^p - s) = 0,\\
1 &+ \bar{u}^p - s = 0,
\end{aligned}$$

so, we have

$$\bar{u} = (s-1)^{\frac{1}{p}},$$

thus

$$\begin{aligned}
\bar{u} &= \frac{\bar{u}}{1+\bar{v}^p},\\
(s-1)^{\frac{1}{p}} &= \frac{(s-1)^{\frac{1}{p}}}{1+\bar{v}^p},\\
(s-1)^{\frac{1}{p}}&(1+\bar{v}^p-1)=0,
\end{aligned}$$

which gives

$$\bar{v}=0.$$

So, we have $(\bar{u},\bar{v}) = \left((s-1)^{\frac{1}{p}},0\right)$.

■

3.4 Boundedness of the positive solution of Fuz. Diff. Eq. (3.1)

Comprehending the bounded behavior of solutions is fundamental, as it guarantees that they remain finite and stable across successive iterations of the system. This property is essential for ensuring that the model produces physically meaningful and realistic results, especially in applications where variables represent quantities such as population levels, concentrations, or resource stocks that cannot grow without limit. From a theoretical standpoint, the boundedness of solutions is closely connected to the overall stability and convergence characteristics of the system, since unbounded trajectories often indicate instability or inconsistency in the underlying dynamics. Furthermore, in numerical simulations or practical implementations, bounded behavior prevents computational overflow and ensures reliable long-term predictions. The next theorem is devoted to establishing sufficient conditions under which the Fuz. Diff.

Eq. (2.1) admits bounded positive solutions. It also provides a systematic framework for identifying and analyzing potential unbounded behaviors within a given range of parameters. This analysis not only strengthens the theoretical understanding of the system's qualitative dynamics but also lays the groundwork for subsequent studies on stability, convergence, and equilibrium behavior.

Theorem 3.4.1 *Let us consider system* (3.13) *under the assumption that* $r, s \in (1, \infty)$. *In this case, the following invariant intervals can be determined for indices* $i \in \{-k, -k+1, \dots, 0\}$.

(i) If $(u_i, v_i) \in]0, (s-1)^{\frac{1}{p}}[\times](r-1)^{\frac{1}{p}}, \infty[$, *then*

$$(u_n, v_n) \in]0, (s-1)^{\frac{1}{p}}[\times](r-1)^{\frac{1}{p}}, \infty[, \quad \textit{for } n \geq 1.$$

(ii) If $(u_i, v_i) \in](s-1)^{\frac{1}{p}}, \infty[\times]0, (r-1)^{\frac{1}{p}}[$, *then*

$$(u_n, v_n) \in](s-1)^{\frac{1}{p}}, \infty[\times]0, (r-1)^{\frac{1}{p}}[, \quad \textit{for } n \geq 1.$$

Proof.

(i) Let $(u_i, v_i) \in \left]0, (s-1)^{1/p}\right[\times \left](r-1)^{\frac{1}{p}}, \infty\right[$ for $i \in \{-k, -k+1, \cdots, 0\}$. So we have

$$u_i < (s-1)^{\frac{1}{p}} = \bar{u}_1$$

and

$$v_i > (r-1)^{\frac{1}{p}} = \bar{v}_1$$

From system (3.13), we obtain

$$\begin{aligned} u_1 &= \frac{ru_{-k}}{1+v_{-1}^p} < \frac{r\bar{u}_1}{1+\bar{v}_1^p} = \bar{u}_1 = (s-1)^{\frac{1}{p}}, \\ v_1 &= \frac{sv_{-k}}{1+u_{-1}^p} > \frac{s\bar{v}_1}{1+\bar{u}_1^p} = \bar{v}_1 = (r-1)^{\frac{1}{p}}. \end{aligned}$$

We prove by induction that

$$(u_n, v_n) \in \Big]0, (s-1)^{1/p}\Big[\times \Big](r-1)^{\frac{1}{p}}, \infty\Big[, \forall n > 1. \tag{3.14}$$

Assume that relation (3.14) holds for some $n = m > 1$. Then, by applying system (3.13), we obtain

$$\begin{aligned} u_{m+1} &= \frac{ru_{m-k}}{1+v_{m-1}^p} < \frac{r\bar{u}_1}{1+\bar{v}_1^p} = \bar{u}_1 = (s-1)^{\frac{1}{p}}, \\ v_{m+1} &= \frac{sv_{m-k}}{1+u_{m-1}^p} > \frac{s\bar{v}_1}{1+\bar{u}_1^p} = \bar{v}_1 = (r-1)^{\frac{1}{p}}. \end{aligned}$$

Hence, the validity of (3.14) extends to every $n \geq -k$, which concludes the proof of part (i).

(ii) Let $(u_i, v_i) \in](s-1)^{\frac{1}{p}}, \infty[\times]0, (r-1)^{\frac{1}{p}}[$ for $i \in \{-k, -k+1, \cdots, 0\}$. So we have

$$u_i > (s-1)^{\frac{1}{p}} = \bar{u}_1$$

and

$$v_i < (r-1)^{\frac{1}{p}} = \bar{v}_1$$

From system (3.13), we obtain

$$\begin{aligned} u_1 &= \frac{ru_{-k}}{1+v_{-1}^p} > \frac{r\bar{u}_1}{1+\bar{v}_1^p} = \bar{u}_1 = (s-1)^{\frac{1}{p}}, \\ v_1 &= \frac{sv_{-k}}{1+u_{-1}^p} < \frac{s\bar{v}_1}{1+\bar{u}_1^p} = \bar{v}_1 = (r-1)^{\frac{1}{p}}. \end{aligned}$$

We prove by induction that

$$(u_n, v_n) \in (u_n, v_n) \in](s-1)^{\frac{1}{p}}, \infty[\times]0, (r-1)^{\frac{1}{p}}[, \forall n > 1. \tag{3.15}$$

Assume that relation (3.15) holds for $n = m+1 > 1$. Then, using system (3.13), we obtain

$$u_{m+1} = \frac{ru_{m-k}}{1+v_{m-1}^p} > \frac{r\bar{u}_1}{1+\bar{v}_1^p} = \bar{u}_1 = (s-1)^{\frac{1}{p}},$$

$$v_{m+1} = \frac{sv_{m-k}}{1+u_{m-1}^p} < \frac{s\bar{v}_1}{1+\bar{u}_1^p} = \bar{v}_1 = (r-1)^{\frac{1}{p}}.$$

Thus, (3.15) remains valid for all $n \geq -k$, which finalizes the proof of case (ii).

■

Theorem 3.4.2 *Assume that $r, s \in]1, +\infty[$. Under this condition, system (3.13) admits unbounded solutions.*

Proof. From Theorem (3.4.1), we obtain

$$u_n < \bar{u}_1 = (s-1)^{1/p}, \qquad v_n > \bar{v}_1 = (r-1)^{1/p}, \quad \text{for } n \geq -k.$$

Therefore,

$$u_{n+1} = \frac{ru_{n-k}}{1+v_{n-1}^p} < \frac{ru_{n-k}}{1+(r-1)} = u_{n-k},$$

and

$$v_{n+1} = \frac{sv_{n-k}}{1+u_{n-1}^p} > \frac{sv_{n-k}}{1+(s-1)} = v_{n-k}.$$

It follows that

$$\lim_{n\to\infty} u_n = 0, \qquad \lim_{n\to\infty} v_n = +\infty. \tag{3.16}$$

This completes the argument. ■

Lemma 3.4.1 *Assume that $r, s \in]0, 1[$. Under this assumption, all positive solutions of equation (3.13) remain bounded and exhibit persistence.*

Proof. For $n > -k$ we have :

$$\begin{aligned}
0 < u_n &= \frac{ru_{n-k-1}}{1+v^p_{n-2}} \\
&< ru_{n-k-1} = r\frac{ru_{n-2(k+1)}}{1+v^p_{n-k-3}} \\
&< r^2 u_{n-2(k+1)} \\
&\vdots \\
&< r^{m+1} u_{n-(m+1)(k+1)}
\end{aligned}$$

For $0 < r < 1$, we obtain:

$$0 < u_n < u_{n-(m+1)(k+1)} \tag{3.17}$$

$$\begin{aligned}
0 < v_n &= \frac{sv_{n-k-1}}{1+u^p_{n-2}} \\
&< sv_{n-k-1} = s\frac{sv_{n-2(k+1)}}{1+u^p_{n-k-3}} \\
&< s^2 v_{n-2(k+1)} \\
&\vdots \\
&< s^{m+1} v_{n-(m+1)(k+1)}
\end{aligned}$$

For $0 < s < 1$, we obtain:

$$0 < v_n < v_{n-(m+1)(k+1)} \tag{3.18}$$

From (3.17) and (3.18):

$$\begin{aligned}
&0 < u_n < u_{-k} \quad if \quad n = (k+1)m + 1 \\
&0 < u_n < u_{-k+1} \quad if \quad n = (k+1)m + 2 \\
&\vdots \\
&0 < u_n < u_{-1} \quad if \quad n = (k+1)m + k \\
&0 < u_n < u_0 \quad if \quad n = (k+1)m + (k+1)
\end{aligned}$$

and

$$
\begin{aligned}
&0 < v_n < v_{-k} \quad if \quad n = (k+1)m + 1\\
&0 < v_n < v_{-k+1} \quad if \quad n = (k+1)m + 2\\
&\vdots\\
&0 < v_n < v_{-1} \quad if \quad n = (k+1)m + k\\
&0 < v_n < v_0 \quad if \quad n = (k+1)m + (k+1)
\end{aligned}
$$

According to the above reasoning, for $n \in \mathbb{N}_0$, $1 \le i \le k+1$, and $n = (k+1)m + i$, the following inequalities hold:

$$0 < u_n < u_{-(k+1)+i}, \qquad 0 < v_n < v_{-(k+1)+i}.$$

This completes the proof. ■

Theorem 3.4.3 *The following assertions hold:*

(i) *If $A_{r,\theta} < 1$ for every $\theta \in]0,1]$, then each positive solution of the Fuz. Diff. Eq. (3.1) is both bounded and persistent.*

(ii) *If there exists some $\bar{\theta} \in]0,1]$ such that $A_{l,\bar{\theta}} > 1$, then the Fuz. Diff. Eq. (3.1) admits unbounded solutions.*

Proof.

(i) Let (ω_n) be a solution of the Fuz. Diff. Eq. (3.1) that satisfies condition (3.4). Using

$$L_{n+1,\theta} = \frac{A_{l,\theta} L_{n-k,\theta}}{1 + R^p_{n-1,\theta}}, \quad R_{n+1,\theta} = \frac{A_{r,\theta} R_{n-k,\theta}}{1 + L^p_{n-1,\theta}},$$

and Lemma (3.4.1), if $A_{r,\theta} = s < 1$ for all $\theta \in]0,1]$ we obtain

$$0 < L_{n,\theta} < R_{n,\theta} < R_{-(k+1)+i,\theta} \quad 1 \le i \le k+1,$$

which gives

$$[L_{n,\theta}, R_{n,\theta}] \subset [0, T_\theta],$$

for $n \in \mathbb{N}_1$, and

$$T_\theta = \max\{R_{-k,\theta}, R_{-k+1,\theta}, \cdots, R_{0,\theta}\}.$$

Since (ω_n) represents a sequence of Pos. Fuz. Nums., there exists a constant $T > 0$ such that $T_\theta \leq T$ for every $\theta \in]0,1]$. Hence, we obtain

$$[L_{n,\theta}, R_{n,\theta}] \subset [0, T], \quad \text{for all } n \in \mathbb{N}_1,$$

which implies

$$\bigcup_{\theta \in]0,1]} [L_{n,\theta}, R_{n,\theta}] \subset [0, T], \quad \text{for all } n \in \mathbb{N}_0.$$

Taking closures, we deduce

$$\overline{\bigcup_{\theta \in]0,1]} [L_{n,\theta}, R_{n,\theta}]} \subseteq [0, T].$$

This completes the proof of case (i).

(ii) Now, suppose there exists some $\bar{\theta} \in]0,1]$ such that $A_{l,\bar{\theta}} > 1$. Set $A_{l,\bar{\theta}} = r$, $A_{r,\bar{\theta}} = s$, $L_{n,\bar{\theta}} = x_n$, and $R_{n,\bar{\theta}} = y_n$ for $n = -k, -k+1, \ldots$. Under this setting, Theorem (3.4.2) can be applied to system (3.3). Thus, if there exists $\bar{\theta} \in]0,1]$ with $r = A_{l,\bar{\theta}} > 1$, then system (3.3) admits solutions (x_n, y_n) corresponding to $\theta = \bar{\theta}$ with initial conditions (x_j, y_j) for $j = -k, -k+1, \ldots, -1, 0$ such that

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty. \tag{3.19}$$

Additionally, if $x_j < y_j$ for $j = -k, -k+1, \cdots, -1, 0$, we can identify Pos. Fuz. Num. ω_j for $j = -k, -k+1, \ldots, -1, 0$, we assume

$$\left[\omega_j\right]_\theta = \left[L_{j,\theta}, R_{j,\theta}\right], \quad \theta \in]0,1], \tag{3.20}$$

and in particular,

$$\left[\omega_j\right]_{\bar{\theta}} = \left[L_{j,\bar{\theta}}, R_{j,\bar{\theta}}\right] = [x_j, y_j], \tag{3.21}$$

for the same indices $j = -k, -k+1, \ldots, -1, 0$.

Let (ω_n) denote a positive solution of equation (3.1) corresponding to the initial values ω_j for $j = -k, -k+1, \ldots, -1, 0$, and $[\omega_n]_\theta = [L_{n,\theta}, R_{n,\theta}]$ for $\theta \in]0, 1]$. Since (3.20) and (3.21) hold and ($L_{n,\theta}, R_{n,\theta}$) satisfies system (3.3) we have

$$[\omega_n]_{\bar{\theta}} = [L_{n,\bar{\theta}}, R_{n,\bar{\theta}}] = [x_n, y_n] \tag{3.22}$$

Thefore, from (3.19), (3.22) and since

$$\|\omega_n\| = \sup \max \left\{ \left| L_{n,\theta} \right|, \left| R_{n,\theta} \right| \right\} \geq \max \left\{ \left| L_{n,\bar{\theta}} \right|, \left| R_{n,\bar{\theta}} \right| \right\} = R_{n,\bar{\theta}},$$

where the supremum is considered over all $\theta \in]0, 1]$. Therefore, the solution (ω_n) turns out to be unbounded.

■

3.5 Convergence of solutions for Fuz. Diff. Eq. (3.1)

In this section, we address the fundamental question of the convergence of positive solutions associated with the Fuz. Diff. Eq. (3.1). Convergence plays a central role in determining both the stability and the asymptotic behavior of such systems, which in turn has direct consequences for their use in applied contexts. By analyzing the convergence characteristics of positive solutions, we aim to identify the conditions that guarantee stable long-term dynamics for equation (3.1). This investigation not only deepens the theoretical understanding of Fuz. Diff. Eqs. but also highlights their relevance and applicability in modeling real-world phenomena where uncertainty and persistence over time are essential considerations.

Theorem 3.5.1 *If $r, s \in]0, 1[$, then the equilibrium point $(0, 0)$ of (3.13) is locally asymptotically stable.*

Proof. Recall that the linearized form of system (3.13) can be expressed as

$$\Psi_{n+1} = J\Psi_n,$$

where

$$\Psi_n = (u_n, u_{n-1}, \ldots, u_{n-k}, v_n, v_{n-1}, \ldots, v_{n-k})^T .$$

The corresponding system, which is equivalent to (3.13), is then represented by

$$\begin{cases} u_{n+1} = \dfrac{ru_{n-k}}{1+v_{n-1}^p}, \\ u_n = \quad u_n, \\ \vdots \\ u_{n-k+1} = \quad u_{n-k+1} \\ v_{n+1} = \dfrac{sv_{n-k}}{1+u_{n-1}^p}, \\ v_n = \quad v_n, \\ \vdots \\ v_{n-k+1} = \quad v_{n-k+1} \end{cases}$$

Thus, the Jacobian matrix evaluated at the equilibrium point $(0,0)$ is expressed as

$$J_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \cdots & r & 0 & 0 & \cdots & 0 \\ 1 & 0 & & & \cdots & & & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & \cdots & 1 & 0 & & \cdots & & 0 \\ 0 & 0 & \cdots & & 0 & & \cdots & s \\ 0 & & \cdots & & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & & & & \cdots & & 1 & 0 \end{pmatrix}.$$

Let $\theta_i, i = 1, 2, \cdots, 2k+2$ be the eigenvalues of J, $\mathcal{L} = \operatorname{diag}(l_1, l_2, \cdots, l_{2k+2})$ be a diagonal matrix, such that:

$$l_1 = l_{k+2} = 1;$$

$$l_{1+i} = l_{k+2+i} = 1 - i\epsilon, \quad 1 \le i \le k$$

$$0 < \epsilon < min\{\frac{1}{k}(1-r), \frac{1}{k}(1-s)\}$$

After some calculations, we obtain

$$\mathcal{L}J\mathcal{L}^{-1}_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \cdots & rl_1 l_{k+1}^{-1} & 0 & & \cdots & 0 \\ l_2 l_1^{-1} & 0 & & & \cdots & & & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & \cdots & l_{k+1} l_k^{-1} & 0 & & & \cdots & 0 \\ 0 & 0 & \cdots & & & & \cdots & sl_{k+2} l_{2k+2}^{-1} \\ 0 & & \cdots & & l_{k+3} l_{k+2}^{-1} & & \cdots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & & & & \cdots & l_{2k+2} l_{2k+1}^{-1} & & 0 \end{pmatrix}.$$

We obtain:

$$0 < l_{k+1} < l_k < \cdots < l_2, \quad 0 < l_{2k+2} < l_{2k+1} < \cdots < l_{k+3},$$

it implies,

$$l_2 l_1^{-1} < 1, \quad l_3 l_2^{-1} < 1, \quad \cdots, \quad l_{k+1} l_k^{-1} < 1$$

$$l_{k+3} l_{k+2}^{-1} < 1, \quad l_{k+4} l_{k+3}^{-1} < 1, \quad \cdots, \quad l_{2k+2} l_{2k+1}^{-1} < 1$$

Furthermore, we have

1. If $min\{\frac{1}{k}(1-r), \frac{1}{k}(1-s)\} = \frac{1}{k}(1-r)$ and $s < r$ then

$$
\begin{aligned}
\epsilon &< \frac{1}{k}(1-r), \\
-k\epsilon &> -k\frac{1}{k}(1-r), \\
1-k\epsilon &> r, \\
\frac{1}{1-k\epsilon} &< \frac{1}{r}, \\
\frac{r}{1-k\epsilon} &< \frac{r}{r}, \\
\frac{r}{1-k\epsilon} &< 1,
\end{aligned}
$$

hence

$$r\, l_1\, l_{k+1}^{-1} < 1.$$

We have also

$$
\begin{aligned}
\epsilon &< \frac{1}{k}(1-r), \\
-k\epsilon &> -k\frac{1}{k}(1-r), \\
1-k\epsilon &> r, \\
\frac{1}{1-k\epsilon} &< \frac{1}{r}, \\
\frac{s}{1-k\epsilon} &< \frac{s}{r}, \\
\frac{s}{1-k\epsilon} &< 1,
\end{aligned}
$$

thus

$$s\, l_{k+2}\, l_{2k+2}^{-1} < 1.$$

2. If $min\{\frac{1}{k}(1-r), \frac{1}{k}(1-s)\} = \frac{1}{k}(1-s)$ and $r < s$ then

$$\begin{aligned}
\epsilon &< \frac{1}{k}(1-s),\\
-k\epsilon &> -k\frac{1}{k}(1-s),\\
1-k\epsilon &> s,\\
\frac{1}{1-k\epsilon} &< \frac{1}{s},\\
\frac{r}{1-k\epsilon} &< \frac{r}{s},\\
\frac{r}{1-k\epsilon} &< 1,
\end{aligned}$$

thus

$$r\, l_1\, l_{k+1}^{-1} < 1.$$

We have also

$$\begin{aligned}
\epsilon &< \frac{1}{k}(1-s),\\
-k\epsilon &> -k\frac{1}{k}(1-s),\\
1-k\epsilon &> s,\\
\frac{1}{1-k\epsilon} &< \frac{1}{s},\\
\frac{s}{1-k\epsilon} &< \frac{s}{s},\\
\frac{s}{1-k\epsilon} &< 1,
\end{aligned}$$

which implies

$$s\, l_{k+2}\, l_{2k+2}^{-1} < 1.$$

$\mathcal{L} J \mathcal{L}^{-1}$ and J have the same eigenvalues

$$\begin{aligned}
\max_{1\le i\le 2k+2}|\theta_i| &\le |\mathcal{L}J\mathcal{L}^{-1}|_\infty \\
&= \max_{1\le j\le 2k+2}\sum_{i=1}^{2k+2}|a_{i,j}| \\
&= \max\{l_2l_1^{-1},\cdots,l_{k+2}l_{k+1}^{-1},\cdots,l_{2k+2}l_{2k+1}^{-1},rl_1l_{k+1}^{-1},sl_{k+2}l_{2k+2}^{-1}\} \\
&< 1
\end{aligned}$$

Therefore, the equilibrium point $(0,0)$ of (3.13) is locally asymptotically stable.

■

Theorem 3.5.2 *If $r,s\in]0,1[$, then every positive solution (u_n,v_n) of* (3.13) *converges to the equilibrium* $(0,0)$.

Proof. It is enough to establish that for any (u_n,v_n) solution of (3.13) that

$$\lim_{n\to\infty}(u_n,v_n)=(u_0,v_0)$$

Since

$$0\le u_{n+1}=\frac{ru_{n-k}}{1+v_{n-1}^p}<ru_{n-k}$$

$$0\le v_{n+1}=\frac{sv_{n-k}}{1+u_{n-1}^p}<sv_{n-k}$$

As $\frac{1}{1+v^p_{(k+1)(n-i)-j}}<1$ and $\frac{1}{1+u^p_{(k+1)(n-i)-j}}<1$, $i=0,1,\cdots,n$ and $j=2,3,\cdots k+2$, then,

$$\begin{aligned}
u_{(k+1)n} &= \frac{ru_{(k+1)(n-1)}}{1+v^p_{(k+1)n-2}} \\
&< r^2u_{(k+1)(n-2)} \\
&< r^3u_{(k+1)(n-3)} \\
&\vdots \\
&< r^nu_0
\end{aligned}$$

$$\begin{aligned} u_{(k+1)n-1} &= \frac{ru_{(k+1)(n-1)-1}}{1+v^p_{(k+1)n-3}} \\ &< r^2 u_{(k+1)(n-2)-1} \\ &< r^3 u_{(k+1)(n-3)-1} \\ &\vdots \\ &< r^n u_{-1} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} u_{(k+1)n-k} &= \frac{ru_{(k+1)(n-1)-k}}{1+v^p_{(k+1)n-k-2}} \\ &< r^2 u_{(k+1)(n-2)-k} \\ &< r^3 u_{(k+1)(n-3)-k} \\ &\vdots \\ &< r^n u_{-k} \end{aligned}$$

Then,

$$\begin{aligned} u_{(k+1)n} &< r^n u_0 \\ u_{(k+1)n-1} &< r^n u_{-1} \\ u_{(k+1)n-2} &< r^n u_{-2} \\ &\vdots \\ u_{(k+1)n-k} &< r^n u_{-k} \end{aligned} \tag{3.23}$$

We have also

$$\begin{aligned} v_{(k+1)n} &= \frac{sv_{(k+1)(n-1)}}{1+u^p_{(k+1)n-2}} \\ &< s^2 v_{(k+1)(n-2)} \\ &< s^3 v_{(k+1)(n-3)} \\ &\vdots \\ &< s^n v_0 \end{aligned}$$

$$\begin{aligned}
v_{(k+1)n-1} &= \frac{sv_{(k+1)(n-1)-1}}{1+u^p_{(k+1)n-3}} \\
&< s^2 v_{(k+1)(n-2)-1} \\
&< s^3 v_{(k+1)(n-3)-1} \\
&\vdots \\
&< s^n v_{-1}
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
v_{(k+1)n-k} &= \frac{sv_{(k+1)(n-1)-1}}{1+u^p_{(k+1)n-k-2}} \\
&< s^2 v_{(k+1)(n-2)-k} \\
&< s^3 v_{(k+1)(n-3)-k} \\
&\vdots \\
&< s^n v_{-k}
\end{aligned}$$

Then,

$$\begin{aligned}
v_{(k+1)n} &< s^n v_0 \\
v_{(k+1)n-1} &< s^n v_{-1} \\
v_{(k+1)n-2} &< s^n v_{-2} \\
&\vdots \\
v_{(k+1)n-k} &< s^n v_{-k}
\end{aligned} \tag{3.24}$$

Thus, for $r < 1$ and $s < 1$, we get

$$\lim_{n\to\infty}(u_n, v_n) = (0,0)$$

■

Theorem 3.5.3 *For the system* (3.13)*, if* $r, s \in]0,1[$*, then the trivial equilibrium* $(0,0)$ *is globally asymptotically stable.*

Theorem 3.5.4 *If $A_{r,\theta} < 1$ for every $\theta \in]0,1]$, then every positive solution of the Fuz. Diff. Eq. (3.1) converges to zero.*

Proof. Assume that equation (3.1) admits a positive solution (ω_n) satisfying condition (3.2), where $A_{r,\theta} < 1$ for all $\theta \in]0,1]$. In this case, Theorem (3.5.2) can be applied to the associated system (3.3). Consequently, we deduce that

$$\lim_{n\to\infty} L_{n,\theta} = \lim_{n\to\infty} R_{n,\theta} = 0. \tag{3.25}$$

Therefore, from (3.25) we get

$$\lim_{n\to\infty} D(\omega_n, 0) = \lim_{n\to\infty} \sup\left\{\max\left\{\left|L_{n,\theta} - 0\right|, \left|R_{n,\theta} - 0\right|\right\}\right\} = 0.$$

■

3.6 Numerical examples

We now present numerical illustrations to complement the theoretical results established in the previous sections concerning the dynamical properties of discrete-time nonlinear higher-order Fuz. Diff. Eqs.. These examples are designed to support and clarify the analytical findings regarding existence, positivity, uniqueness, and qualitative behavior of the solutions. By means of numerical simulations, we demonstrate the boundedness, persistence, and convergence of Pos. Fuz. Sols., thereby reinforcing the theoretical framework. Such simulations also highlight the practical relevance and applicability of the obtained results.

Let us consider the Fuz. Diff. Eq. (3.1), where ω_n represents a sequence of Pos. Fuz. Nums. and $A \in \mathbb{R}_{\mathcal{F}}^{+}$. In this setting, we take $k = 3$ and $p = 2$, with the initial values $\omega_i \in \mathbb{R}_{\mathcal{F}}^{+}$ for $i \in \{-3, -2, -1, 0\}$ given as follows:

$$\omega_{-3}(x) = \begin{cases} 4x-6, & \frac{3}{2} \le x \le \frac{7}{4} \\ -4x+8, & \frac{7}{4} \le x \le 2 \end{cases}, \omega_{-2}(x) = \begin{cases} \frac{1}{3}x-\frac{3}{2}, & \frac{9}{2} \le x \le 6 \\ -\frac{1}{4}x-2, & 6 \le x \le 8 \end{cases}$$
$$\omega_{-1}(x) = \begin{cases} \frac{1}{4}x-\frac{1}{4}, & 1 \le x \le 5 \\ -\frac{1}{4}x+2, & 5 \le x \le 8 \end{cases}, \omega_{0}(x) = \begin{cases} x-3, & 3 \le x \le 4 \\ -x+5, & 4 \le x \le 5 \end{cases} \tag{3.26}$$

We have

$$[\omega_{-3}]_\theta = \left[\frac{1}{4}\theta + \frac{6}{4}, 2 - \frac{1}{4}\theta\right], \quad [\omega_{-2}]_\theta = \left[\frac{9}{2} + 3\theta, 8 - 4\theta\right], \quad \theta \in]0,1]$$
$$[\omega_{-1}]_\theta = [1+4\theta, 8-4\theta], \quad [\omega_0]_\theta = [3+\theta, 5-\theta],$$

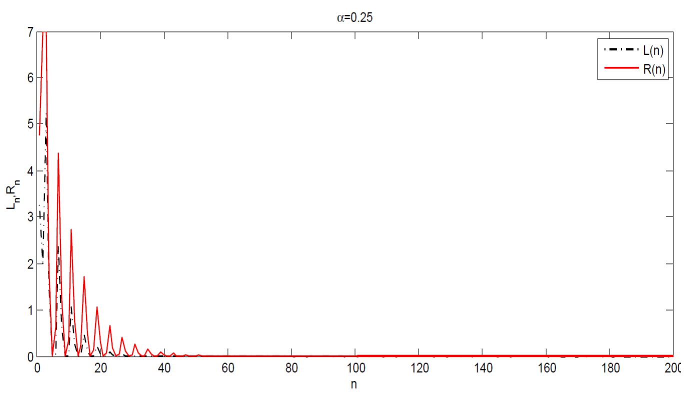
Example 3.6.1 *In this numerical illustration, the condition on A is fulfilled.*

$$A(x) = \begin{cases} 5x-2, & \frac{2}{5} \le x \le \frac{1}{2} \\ 4-6x, & \frac{1}{2} \le x \le \frac{4}{6} \end{cases}$$

Therfore, it follows that

$$[A]_\theta = \left[\frac{\theta+2}{5}, \frac{4-\theta}{6}\right] \tag{3.27}$$

From (3.27), it follows that $A_{r,\theta} < 1$ for every $\theta \in]0,1]$. Hence, applying Theorem (3.5.4), we conclude that the positive solution ω_n of equation (3.1) converges to 0 as $n \to \infty$.



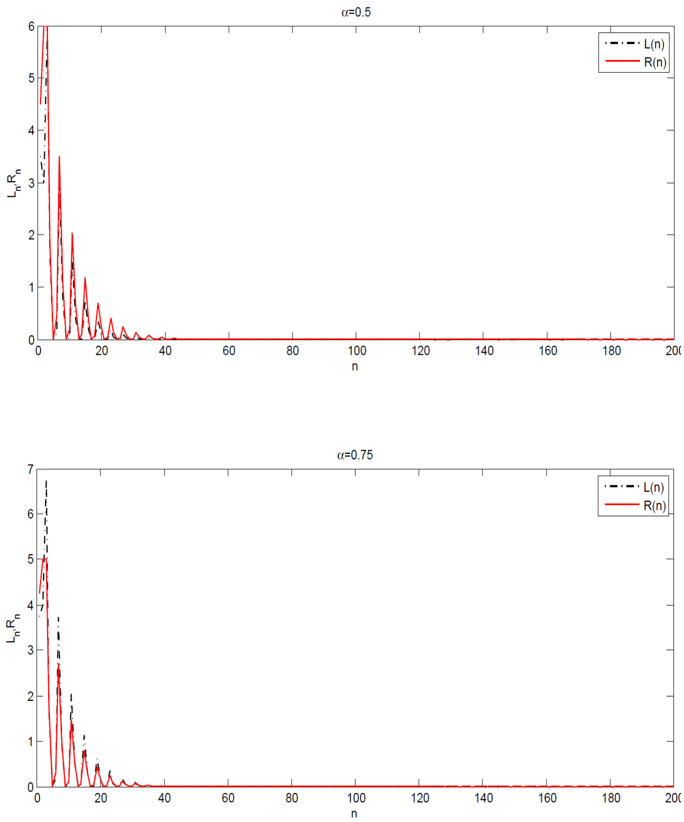


Figure 3.1: The solution of the system (3.1) at $\theta = 0.25$, $\theta = 0.5$ and $\theta = 0.75$

Example 3.6.2 *In this example, the condition on A is fulfilled.*

$$A(x) = \begin{cases} x - 4, & 4 \leq x \leq 5 \\ 6 - x, & 5 \leq x \leq 6 \end{cases}$$

We have

$$[A]_\theta = [\theta + 4, 6 - \theta] \quad \theta \in]0, 1] \tag{3.28}$$

Based on Theorem (3.2.1), equation (3.1) admits a unique positive solution. Moreover, it is clear that $A_{l,\theta} > 1$ for every $\theta \in]0, 1]$. Therefore, by applying case (ii) of Theorem (3.4.3), we deduce that equation (3.1) possesses unbounded solutions.

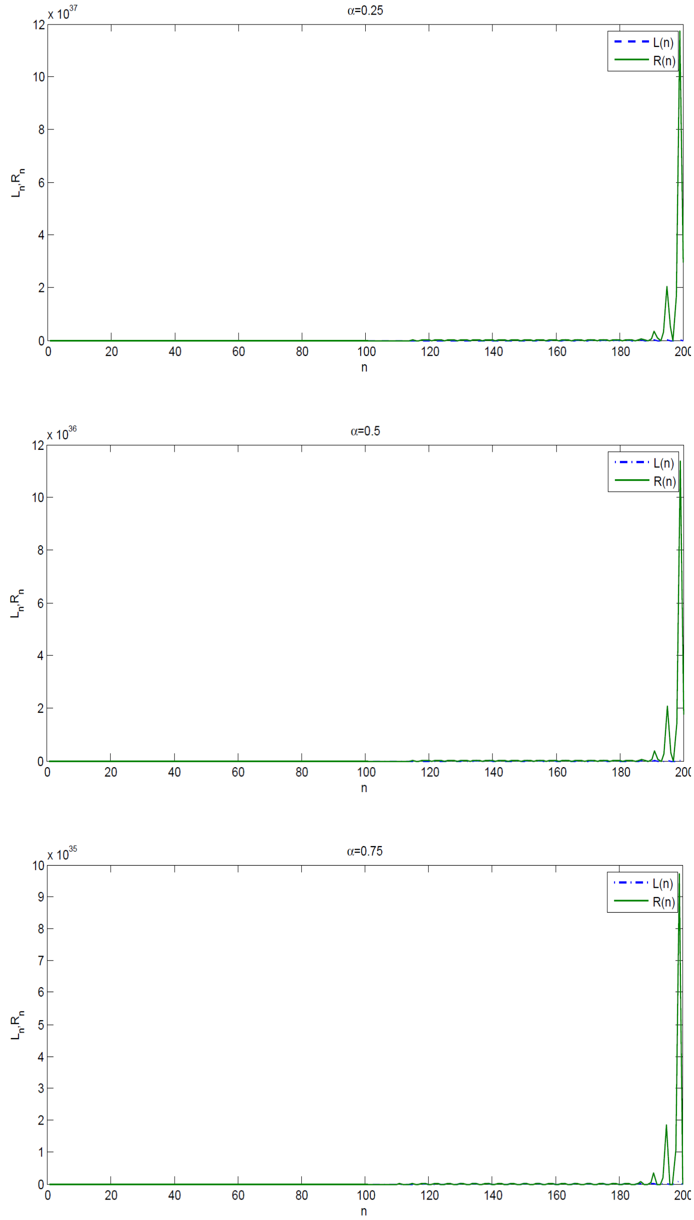


Figure 3.2: The solution of the system (3.1) at $\theta = 0.25$, $\theta = 0.5$ and $\theta = 0.75$

3.7 Conclusion

Chapter 3 extends the qualitative analysis developed in the previous chapter to a broader and more general class of fuzzy difference equations of power–type. The main object of study is a higher–order nonlinear fuzzy recurrence of the form

$$y_{n+1} = \beta + \frac{\gamma\, y_n^p}{y_{n-\kappa}^q}, \qquad n = 0, 1, \ldots, \qquad \kappa \geq 2, \quad p, q > 0,$$

where β, γ, and the initial values $y_{-\kappa}, \ldots, y_0$ are positive fuzzy numbers. This equation generalizes the model studied in Chapter 2 by replacing the linear term y_n and the quadratic denominator with power–type expressions. Such a generalization allows the chapter to capture a richer variety of nonlinear behaviors and to cover several important fuzzy models appearing in population dynamics, control systems, and uncertain biological processes.

The chapter begins by establishing the well-posedness of the recurrence. Using the θ-cut approach, the fuzzy equation is decomposed into a system of crisp interval recurrences whose solutions correspond to the θ-level representations of the fuzzy solution. By exploiting the monotonicity of the functions $x \mapsto x^p$ and $x \mapsto x^{-q}$ on positive intervals, the chapter derives sufficient conditions ensuring existence and uniqueness of a positive fuzzy solution for all $n \geq 0$. The analysis demonstrates that the generalized power-type formulation does not disrupt the structural properties established in Chapter 2, although its nonlinearities require more delicate interval estimates.

A major portion of the chapter is dedicated to the boundedness and persistence of the fuzzy solutions. Through careful examination of the lower and upper θ-cut sequences, the chapter provides conditions guaranteeing that all solutions remain in fixed invariant fuzzy intervals. The parameter exponents p and q play a crucial role in shaping the dynamics; in particular, the relative sizes of p and q determine whether the numerator or the denominator dominates as n becomes large. The chapter shows that for a broad range of parameter values, the fuzzy solutions avoid extinction and unbounded growth, and instead remain confined within intervals that depend explicitly on β, γ, p, q, and the initial fuzzy data.

The chapter then investigates the equilibrium structure of the recurrence. By solving the algebraic equation

$$\bar{y} = \beta + \frac{\gamma \bar{y}^{p}}{\bar{y}^{q}},$$

it is shown that under mild assumptions there exists a unique positive equilibrium $\bar{y}$. The θ-cut formulation transforms the convergence problem into a pair of interval recurrences, which are then studied using classical tools from nonlinear difference equations. The chapter establishes sufficient conditions for global convergence of every positive fuzzy solution to the fuzzy equilibrium. Depending on the exponents p and q, the convergence may be monotone or oscillatory. A key feature of the proof is the preservation of the ordering of interval endpoints across successive iterations, which ensures that convergence at each θ level implies convergence of the entire fuzzy sequence.

An additional contribution of the chapter is the analysis of the rate of convergence. The behavior of the mapping near the equilibrium is studied by examining the derivative-like quantities associated with the interval endpoints. It is demonstrated that for certain relations between p and q, the convergence is geometric, while in other cases it may be slower and depend more sensitively on the initial fuzzy conditions. This part reveals subtle interactions between the degrees of fuzziness and the nonlinear power structure of the equation.

The chapter concludes with numerical simulations that illustrate the theoretical results. Several examples are presented for various choices of exponents p and q, highlighting how the power-type nonlinearity influences the width and position of the fuzzy θ-cuts over time. The numerical experiments show that fuzziness tends to contract around the equilibrium under stable parameter settings, in full agreement with the analytical predictions.

Overall, Chapter 3 represents a significant generalization of the results from Chapter 2. It broadens the scope of the qualitative theory by accommodating a wide range of nonlinearities, introduces advanced analytical tools for handling fuzzy power-type recurrences, and sets the stage for Chapter 4, where multidimensional systems of nonlinear fuzzy difference equations are investigated. Thus, the chapter serves both as an extension of earlier results and as a bridge to the more complex dynamical models studied in the final part of the dissertation.

Chapter 4

Dynamics of a higher-order three-dimensional rational difference system

4.1 Introduction

Difference equations constitute an important tool in the study of discrete dynamical systems, with broad applications in mathematical biology, economics, and engineering. Classical references such as Jones and Sleeman [40] and Lakshmikantham and Trigiante [55] laid the foundational aspects of difference equations and their role in applied sciences. Later, several numerical and analytical techniques were developed to handle nonlocal or higher-dimensional structures, as illustrated for instance in Wang et al. [69], Zhou et al. [81], and the recent contribution of Zhang et al. [82].

In the context of nonlinear rational systems, numerous investigations have been carried out to understand stability, boundedness, persistence, and oscillatory behavior. Important contributions include the works of Mesmouli et al. [56] on nonlinear systems with delays, as well as Gumus [26, 27, 28] and Gumus & Soykan [29], where global stability and periodicity properties are analyzed. The research line was further enriched by Elsayed and co-authors [18, 19, 20], who provided systematic studies on periodicity and rational forms of difference equations.

Another recent trend is the investigation of high-order and multidimensional systems. The works of Kara and collaborators [46, 47, 48], Yazlik and co-authors [72, 73], as well as Khelifa and Halim [49], have demonstrated the complexity and richness of such systems. In parallel, Halim and collaborators [31, 35, 30] have established results on the dynamics of higher-order and multidimensional systems, including connections with generalized Fibonacci sequences.

These diverse contributions show that rational difference equations continue to be a fertile field of research, with ongoing efforts to generalize classical results to more complex systems involving multiple delays, higher dimensions, and nonlinearities. The present study follows this line of inquiry and develops an alternative extension that complements and extends previous works.

Recently, [57], [82] and [3] established results on persistence, periodicity, and global stability of their systems.

Motivated by these contributions, the present paper develops a complementary generalization of the work of Zhang et al.[82], *in the direction of delay*. Specifically, we consider the delayed three-dimensional system

$$\begin{cases} \eta_{n+1} = \phi + \dfrac{\eta_{n-m}^{q}}{\omega_n^{q}}, \\ \omega_{n+1} = \phi + \dfrac{\omega_{n-m}^{q}}{\xi_n^{q}}, \\ \xi_{n+1} = \phi + \dfrac{\xi_{n-m}^{q}}{\eta_n^{q}}, \end{cases} \tag{4.1}$$

where $q \in [1, \infty)$, $\phi > 0$, and the initial conditions are strictly positive. Our objective is to investigate the dynamics of this system, with particular emphasis on stability, periodicity, and oscillatory behavior, and to highlight the role of the delay parameter m together with the nonlinear exponent q in shaping the system?s global dynamics.

4.2 Local stability of solutions

A significant finding of this investigation concerning the system?s qualitative behavior is presented below. In particular, we rigorously determine the precise circumstances under which the unique positive equilibrium exhibits local asymptotic stability or, conversely, becomes unstable. This distinction plays a crucial role in understanding how small perturbations around the equilibrium evolve over time and whether the trajectories tend to return to equilibrium or diverge from it. The identification of these stability conditions not only enhances our theoretical insight into the underlying dynamics but also provides a solid framework for interpreting the system?s long-term behavior. Consequently, this result constitutes a key analytical foundation upon which the subsequent sections build, guiding both the mathematical analysis and the numerical exploration that follow.

Theorem 4.2.1 *The system* (4.1) *has a positive equilibrium* $(\bar{\eta}, \bar{\omega}, \bar{\xi}) = (\phi + 1, \phi + 1, \phi + 1)$ *that is locally asymptotically stable whenever* $\phi > 2q - 1$. *If the condition* $0 < \phi < 2q - 1$ *is met, the equilibrium becomes unstable.*

Proof. Let η, ω, ξ be positive real numbers satisfying

$$\begin{cases} \eta = \phi + \dfrac{\eta^q}{\omega^q}, \\ \omega = \phi + \dfrac{\omega^q}{\xi^q}, \\ \xi = \phi + \dfrac{\xi^q}{\eta^q}, \end{cases}$$

then the system admits a positive equilibrium given by $(\bar{\eta}, \bar{\omega}, \bar{\xi}) = (\phi + 1, \phi + 1, \phi + 1)$.

The linearized form of system (4.1) around the positive equilibrium point $(\phi + 1, \phi + 1, \phi + 1)$ is given by

$$U_{n+1} = GU_n,$$

where

$$U_n = (\eta_n, \eta_{n-1}, \ldots, \eta_{n-m}, \omega_n, \omega_{n-1}, \ldots, \omega_{n-m}, \xi_n, \xi_{n-1}, \ldots, \xi_{n-m})^T$$

and G is the Jacobian matrix of the linearized system evaluated at the equilibrium. It is expressed as:

$$G = \begin{pmatrix}
0 & 0 & \cdots & \frac{q}{\phi+1} & -\frac{q}{\phi+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{q}{\phi+1} & -\frac{q}{\phi+1} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{q}{\phi+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{q}{\phi+1} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & & & \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.$$

Let λ_i for $i \in \{1, \ldots, 3m+3\}$ denote the eigenvalues of G, and define a diagonal matrix $L = \mathrm{diag}(l_1, \ldots, l_{3m+3})$ with:

$$l_1 = l_{m+2} = l_{2m+3} = 1, \quad l_i = l_{m+1+i} = l_{2m+2+i} = 1 - i\varepsilon, \quad 2 \le i \le m+1,$$

where

$$0 < \varepsilon < \frac{1}{3m+3}\left(\frac{\phi+1-2q}{\phi+1-q}\right). \tag{4.2}$$

As L is invertible, one computes LGL^{-1} as follows:

$$LGL^{-1} = \begin{pmatrix}
0 & 0 & \cdots & \frac{ql_1 l_{m+1}^{-1}}{\phi+1} & -\frac{ql_1 l_{m+2}^{-1}}{\phi+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
l_2 l_1^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & \\
0 & 0 & \cdots & l_{m+1} l_m^{-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & \frac{ql_{m+2} l_{2m+2}^{-1}}{\phi+1} & -\frac{ql_{m+2} l_{2m+3}^{-1}}{\phi+1} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
-\frac{ql_{2m+3} l_1^{-1}}{\phi+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{ql_{2m+3} l_{3m+3}^{-1}}{\phi+1} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & l_{2m+4} l_{2m+3}^{-1} & \cdots & 0 & 0 \\
\vdots & & & & & & & & & & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & l_{3m+3} l_{3m+2}^{-1} & 0
\end{pmatrix}$$

Note that:

$$l_1 > l_2 > \cdots > l_{m+1}, \quad l_{m+2} > \cdots > l_{2m+2}, \quad l_{2m+3} > \cdots > l_{3m+3},$$

which leads to the inequalities:

$l_2 l_1^{-1} < 1$, $l_3 l_2^{-1} < 1$, $\cdots$, $l_{m+1} l_m^{-1} < 1$, $l_{m+1} l_m^{-1} < 1$ $l_{m+3} l_{m+2}^{-1} < 1$, $\cdots$, $l_{2m+2} l_{2m+1}^{-1} < 1$, $l_{2m+4} l_{2m+3}^{-1} < 1$, $\cdots$, $l_{3m+3} l_{3m+2}^{-1} < 1$

Moreover, by (4.2), we obtain:

$$\begin{aligned}
\frac{ql_1 L_{m+1}^{-1}}{\phi+1} + \frac{ql_1 l_{m+2}^{-1}}{\phi+1} &= \frac{q}{\phi+1}\left(1 + \frac{1}{l_{m+1}}\right) = \frac{q}{\phi+1}\left(1 + \frac{1}{1-(m+1)\varepsilon}\right) < 1, \\
\frac{ql_{m+2} l_{2m+2}^{-1}}{\phi+1} + \frac{ql_{m+2} l_{2m+3}^{-1}}{\phi+1} &= \frac{q}{\phi+1}\left(1 + \frac{1}{l_{2m+2}}\right) = \frac{q}{\phi+1}\left(1 + \frac{1}{1-(2m+2)\varepsilon}\right) < 1, \\
\frac{ql_{2m+3} l_1^{-1}}{\phi+1} + \frac{ql_{2m+3} l_{3m+3}^{-1}}{\phi+1} &= \frac{q}{\phi+1}\left(1 + \frac{1}{l_{3m+3}}\right) = \frac{q}{\phi+1}\left(1 + \frac{1}{1-(3m+3)\varepsilon}\right) < 1.
\end{aligned}$$

Since the eigenvalues of G and LGL^{-1} are the same, it follows that:

$$\begin{aligned}
\max_{1\leq i\leq 3m+3} |\lambda| &\leq \left\| LGL^{-1} \right\| \\
&= \max\Big\{ l_2 l_1^{-1},\ l_3 l_2^{-1},\ \ldots,\ l_{m+1} l_m^{-1},\ l_{m+3} l_{m+2}^{-1},\ \ldots, \\
&\qquad l_{2m+2} l_{2m+1}^{-1},\ l_{2m+4} l_{2m+3}^{-1},\ \ldots,\ l_{3m+3} l_{3m+2}^{-1}, \\
&\qquad \frac{q}{\phi+1}\left(1+\frac{1}{l_{m+1}}\right),\ \frac{q}{\phi+1}\left(1+\frac{1}{l_{2m+2}}\right),\ \frac{q}{\phi+1}\left(1+\frac{1}{l_{3m+3}}\right)\Big\} \\
&< 1.
\end{aligned}$$

Hence, the equilibrium point $(\phi+1, \phi+1, \phi+1)$ of system (4.1) is locally asymptotically stable.

We conclude that the instability of the equilibrium under the condition $0 < \phi < 2q-1$ follows directly from the preceding arguments established in the proof.

■

4.3 Asymptotic behavior of solutions

Following the local stability result established in Theorem 4.2.1, we proceed to conduct a deeper investigation into the asymptotic behavior of the system's solutions under a set of additional constraints. In particular, we focus on the case where the parameter ϕ lies within the open interval $]0,1[$ and the delay parameter m takes odd integer values. These specific conditions introduce notable qualitative changes in the system's evolution, influencing both the nature and stability of the trajectories over time. As will be demonstrated in the subsequent theorem, such parameter configurations give rise to distinct long-term dynamical patterns, offering valuable insight into how the delay and nonlinearity interact to shape the overall asymptotic profile of the system.

Theorem 4.3.1 *Let $\phi \in]0,1[$, m be an odd integer, and suppose that $\{(\eta_n, \omega_n, \xi_n)\}$ is a positive solution of system* (4.1)*. Then the following statements are valid:*

(i) If

$$0 < \eta_{2i-1} < 1, \quad 0 < \omega_{2i-1} < 1, \quad 0 < \xi_{2i-1} < 1$$

and

$$\eta_{2i} > \frac{1}{(1-\phi)^{1/q}}, \quad \omega_{2i} > \frac{1}{(1-\phi)^{1/q}}, \quad \xi_{2i} > \frac{1}{(1-\phi)^{1/q}}$$

for indices $i = \frac{1-m}{2}, \frac{3-m}{2}, \cdots, 0$, *then*

$$\lim_{n\to\infty} \eta_{2n} = \lim_{n\to\infty} \omega_{2n} = \lim_{n\to\infty} \xi_{2n} = \infty$$

and

$$\lim_{n\to\infty} \eta_{2n+1} = \lim_{n\to\infty} \omega_{2n+1} = \lim_{n\to\infty} \xi_{2n+1} = \phi.$$

(ii) If

$$\eta_{2i-1} > \frac{1}{(1-\phi)^{1/q}}, \quad \omega_{2i-1} > \frac{1}{(1-\phi)^{1/q}}, \quad \xi_{2i-1} > \frac{1}{(1-\phi)^{1/q}}$$

and

$$0 < \eta_{2i} < 1, \quad 0 < \omega_{2i} < 1, \quad 0 < \xi_{2i} < 1$$

for indices $i = \frac{1-m}{2}, \frac{3-m}{2}, \cdots, 0$, *then*

$$\lim_{n\to\infty} \eta_{2n} = \lim_{n\to\infty} \omega_{2n} = \lim_{n\to\infty} \xi_{2n} = \phi$$

and

$$\lim_{n\to\infty} \eta_{2n+1} = \lim_{n\to\infty} \omega_{2n+1} = \lim_{n\to\infty} \xi_{2n+1} = \infty.$$

Proof.

(i) We start by noting that

$$\begin{cases} \phi < \eta_1 = \phi + \dfrac{\eta_{-m}^q}{\omega_0^q} \le \phi + \dfrac{1}{\omega_0^q} \le \phi + 1 - \phi = 1, \\ \phi < \omega_1 = \phi + \dfrac{\omega_{-m}^q}{\xi_0^q} \le \phi + \dfrac{1}{\xi_0^q} \le \phi + 1 - \phi = 1, \\ \phi < \xi_1 = \phi + \dfrac{\xi_{-m}^q}{\eta_0^q} \le \phi + \dfrac{1}{\eta_0^q} \le \phi + 1 - \phi = 1. \end{cases}$$

This shows that

$$(\eta_1, \omega_1, \xi_1) \in]\phi, 1] \times]\phi, 1] \times]\phi, 1].$$

Similarly, we obtain

$$\begin{cases} \eta_2 = \phi + \dfrac{\eta^q_{-m+1}}{\omega^q_1} \geq \phi + \eta^q_{-m+1} > \phi + \dfrac{1}{1-\phi}, \\ \omega_2 = \phi + \dfrac{\omega^q_{-m+1}}{\xi^q_1} \geq \phi + \omega^q_{-m+1} > \phi + \dfrac{1}{1-\phi}, \\ \xi_2 = \phi + \dfrac{\xi^q_{-m+1}}{\eta^q_1} \geq \phi + \xi^q_{-m+1} > \phi + \dfrac{1}{1-\phi}. \end{cases}$$

Hence,

$$(\eta_2, \omega_2, \xi_2) \in [\phi + \tfrac{1}{1-\phi}, +\infty[\times [\phi + \tfrac{1}{1-\phi}, +\infty[\times [\phi + \tfrac{1}{1-\phi}, +\infty[.$$

Proceeding in the same way, we have

$$\begin{cases} \phi < \eta_3 = \phi + \dfrac{\eta^q_{-m+2}}{\omega^q_2} \leq \phi + \dfrac{1}{\omega^q_2} \leq \phi + 1 - \phi = 1, \\ \phi < \omega_3 = \phi + \dfrac{\omega^q_{-m+2}}{\xi^q_2} \leq \phi + \dfrac{1}{\xi^q_2} \leq \phi + 1 - \phi = 1, \\ \phi < \xi_3 = \phi + \dfrac{\xi^q_{-m+2}}{\eta^q_2} \leq \phi + \dfrac{1}{\eta^q_2} \leq \phi + 1 - \phi = 1. \end{cases}$$

Which leads to

$$(\eta_3, \omega_3, \xi_3) \in]\phi, 1] \times]\phi, 1] \times]\phi, 1].$$

Likewise, we get

$$\begin{cases} \eta_4 = \phi + \dfrac{\eta^q_{-m+3}}{\omega^q_3} \geq \phi + \eta^q_{-m+3} > \phi + \dfrac{1}{1-\phi}, \\ \omega_4 = \phi + \dfrac{\omega^q_{-m+3}}{\xi^q_3} \geq \phi + \omega^q_{-m+3} > \phi + \dfrac{1}{1-\phi}, \\ \xi_4 = \phi + \dfrac{\xi^q_{-m+3}}{\eta^q_3} \geq \phi + \xi^q_{-m+3} > \phi + \dfrac{1}{1-\phi}. \end{cases}$$

$$\vdots$$

By induction, for any integer $l \geq 1$, we have

$$\begin{cases} \eta_{2l} = \phi + \dfrac{\eta_{2l-m-1}^q}{\omega_{2l-1}^q} \geq \phi + \eta_{2l-m-1}^q > \phi + \dfrac{1}{1-\phi}, \\ \omega_{2l} = \phi + \dfrac{\omega_{2l-m-1}^q}{\xi_{2l-1}^q} \geq \phi + \omega_{2l-m-1}^q > \phi + \dfrac{1}{1-\phi}, \\ \xi_{2l} = \phi + \dfrac{\xi_{2l-m-1}^q}{\eta_{2l-1}^q} \geq \phi + \xi_{2l-m-1}^q > \phi + \dfrac{1}{1-\phi}. \end{cases}$$

And for even larger indices,

$$\begin{cases} \eta_{4l} = \phi + \dfrac{\eta_{4l-m-1}^q}{\omega_{4l-1}^q} \geq \phi + \eta_{4l-m-1}^q > \phi + \dfrac{1}{1-\phi}, \\ \omega_{4l} = \phi + \dfrac{\omega_{4l-m-1}^q}{\xi_{4l-1}^q} \geq \phi + \omega_{4l-m-1}^q > \phi + \dfrac{1}{1-\phi}, \\ \xi_{4l} = \phi + \dfrac{\xi_{4l-m-1}^q}{\eta_{4l-1}^q} \geq \phi + \xi_{4l-m-1}^q > \phi + \dfrac{1}{1-\phi}. \end{cases}$$

$$\vdots$$

Eventually, for every $p \in \mathbb{N}$,

$$\begin{cases} \eta_{2pl} = \phi + \dfrac{\eta_{2pl-m-1}^q}{\omega_{2pl-1}^q} \geq \phi + \eta_{2pl-m-1}^q > \phi + \dfrac{1}{1-\phi}, \\ \omega_{2pl} = \phi + \dfrac{\omega_{2pl-m-1}^q}{\xi_{2pl-1}^q} \geq \phi + \omega_{2pl-m-1}^q > \phi + \dfrac{1}{1-\phi}, \\ \xi_{2pl} = \phi + \dfrac{\xi_{2pl-m-1}^q}{\eta_{2pl-1}^q} \geq \phi + \xi_{2pl-m-1}^q > \phi + \dfrac{1}{1-\phi}. \end{cases}$$

Thus, if $n = pl$, we obtain:

$$\lim_{n\to\infty} \eta_{2n} = \infty, \quad \lim_{n\to\infty} \omega_{2n} = \infty, \quad \lim_{n\to\infty} \xi_{2n} = \infty.$$

Now, applying the assumption in (i) and passing to the limit on both sides of each equation in (4.1):

$$\eta_{2n+1} = \phi + \frac{\eta^q_{2n-m}}{\omega^q_{2n}}, \quad \omega_{2n+1} = \phi + \frac{\omega^q_{2n-m}}{\xi^q_{2n}}, \quad \xi_{2n+1} = \phi + \frac{\xi^q_{2n-m}}{\eta^q_{2n}}.$$

Taking limits yields:

$$\lim_{n\to\infty} \eta_{2n+1} = \lim_{n\to\infty} \omega_{2n+1} = \lim_{n\to\infty} \xi_{2n+1} = \phi.$$

(ii) The proof of (ii) follows in the same way and is therefore omitted.

■

4.4 Oscillation of solutions

After establishing in Theorem 4.3.1 the asymptotic properties of positive solutions, our next objective is to explore their oscillatory behavior in greater depth. Understanding whether the solutions oscillate around the equilibrium is crucial for capturing the complete dynamical portrait of the system, as oscillations often reflect complex interactions between the parameters and delayed terms. The forthcoming result formulates explicit and sufficient conditions ensuring that every component of the solution exhibits oscillatory motion about the equilibrium point. This analysis not only enriches the qualitative description of the system's dynamics but also highlights the delicate balance between stability and periodic fluctuation inherent in such nonlinear delayed structures.

Theorem 4.4.1 *Suppose that $\{(\eta_n, \omega_n, \xi_n)\}$ is a positive solution of system (4.1), and let m be an odd number. If one of the following conditions holds for some $n_0 \geq 0$, then all components of the solution are oscillatory.*

(i)

$$\begin{cases} \eta_{n_0}, \eta_{n_0+2}, \cdots, \eta_{n_0+m-1} < \phi + 1 \leq \eta_{n_0+1}, \eta_{n_0+3}, \cdots, \eta_{n_0+m} \\ \omega_{n_0}, \omega_{n_0+2}, \cdots, \omega_{n_0+m-1} < \phi + 1 \leq \omega_{n_0+1}, \omega_{n_0+3}, \cdots, \omega_{n_0+m} \\ \xi_{n_0}, \xi_{n_0+2}, \cdots, \xi_{n_0+m-1} < \phi + 1 \leq \xi_{n_0+1}, \xi_{n_0+3}, \cdots, \xi_{n_0+m} \end{cases}$$

(ii)

$$\begin{cases} \eta_{n_0}, \eta_{n_0+2}, \cdots, \eta_{n_0+m-1} > \phi + 1 \geq \eta_{n_0+1}, \eta_{n_0+3}, \cdots, \eta_{n_0+m} \\ \omega_{n_0}, \omega_{n_0+2}, \cdots, \omega_{n_0+m-1} > \phi + 1 \geq \omega_{n_0+1}, \omega_{n_0+3}, \cdots, \omega_{n_0+m} \\ \xi_{n_0}, \xi_{n_0+2}, \cdots, \xi_{n_0+m-1} > \phi + 1 \geq \xi_{n_0+1}, \xi_{n_0+3}, \cdots, \xi_{n_0+m} \end{cases}$$

Proof.

(i) If condition (*i*) is satisfied, then:

$$\begin{cases} \eta_{n_0}^q < (\phi+1)^q \quad \text{and} \quad \dfrac{1}{\omega_{n_0+m}^q} \leq \dfrac{1}{(\phi+1)^q}, \\ \omega_{n_0}^q < (\phi+1)^q \quad \text{and} \quad \dfrac{1}{\xi_{n_0+m}^q} \leq \dfrac{1}{(\phi+1)^q}, \\ \xi_{n_0}^q < (\phi+1)^q \quad \text{and} \quad \dfrac{1}{\eta_{n_0+m}^q} \leq \dfrac{1}{(\phi+1)^q}. \end{cases}$$

It follows that:

$$\begin{cases} \eta_{n_0+m+1} < \phi + 1, \\ \omega_{n_0+m+1} < \phi + 1, \\ \xi_{n_0+m+1} < \phi + 1. \end{cases}$$

Moreover, we also have:

$$\begin{cases} \eta_{n_0+1}^q \geq (\phi+1)^q \quad \text{and} \quad \dfrac{1}{\omega_{n_0+m+1}^q} > \dfrac{1}{(\phi+1)^q}, \\ \omega_{n_0+1}^q \geq (\phi+1)^q \quad \text{and} \quad \dfrac{1}{\xi_{n_0+m+1}^q} > \dfrac{1}{(\phi+1)^q}, \\ \xi_{n_0+1}^q \geq (\phi+1)^q \quad \text{and} \quad \dfrac{1}{\eta_{n_0+m+1}^q} > \dfrac{1}{(\phi+1)^q}. \end{cases}$$

Hence:

$$\begin{cases} \eta_{n_0+m+2} > \phi + 1, \\ \omega_{n_0+m+2} > \phi + 1, \\ \xi_{n_0+m+2} > \phi + 1. \end{cases}$$

As a result, we obtain:

$$\eta_{n_0+m+1} < \phi + 1 < \eta_{n_0+m+2}; \quad \omega_{n_0+m+1} < \phi + 1 < \omega_{n_0+m+2}; \quad \xi_{n_0+m+1} < \phi + 1 < \xi_{n_0+m+2}$$

(ii) The same reasoning applies in the case where condition (*ii*) is satisfied.

■

4.5 Persistence of solutions

Persistence represents a cornerstone concept in the theory of dynamical systems, signifying that the variables of interest remain uniformly bounded away from both zero and infinity as time progresses. This property guarantees that the system evolves within a biologically, physically, or economically meaningful range, thereby ensuring its long-term stability and viability. From a practical standpoint, persistence serves as a safeguard against extinction, explosion, or degeneracy in the modeled process. In biological or ecological frameworks, it reflects the capacity of populations or species to survive indefinitely without declining to extinction, while in economic or technological models, it prevents essential quantities such as capital, production, or resources from collapsing to zero or diverging uncontrollably. Consequently, establishing persistence provides an essential foundation for a deeper understanding of equilibrium behavior and sustainable dynamics within complex systems.

Definition 4.5.1 *We say that system* (4.1) *is persistent if there are positive constants m and M*

such that every positive solution $\{(\eta_n, \omega_n, \xi_n)\}$ *to the system satisfies the following inequalities*

$$m \le \eta_n, \omega_n, \xi_n \le M,$$

for sufficiently large n.

This result demonstrates that, under a certain condition on the parameter ϕ, the system's solutions are both bounded and persistent. It guarantees that the solutions remain positive and avoid divergence.

Theorem 4.5.1 *Assume that* $\phi^q > 1$. *Then, every positive solution* $\{(\eta_n, \omega_n, \xi_n)\}$ *of system* (4.1) *remains bounded and persists.*

Proof. From system (4.1), for all $n \ge 1$, it follows that

$$\eta_n \ge \phi, \quad \omega_n \ge \phi, \quad \xi_n \ge \phi. \tag{4.3}$$

Hence, we get:

$$\eta_n = \phi + \frac{\eta_{n-m-1}^q}{\omega_{n-1}^q} \le \phi + \frac{\eta_{n-m-1}^q}{\phi^q}.$$

Moreover, since

$$\eta_{n-m-1} = \phi + \frac{\eta_{n-2(m+1)}^q}{\omega_{n-m-2}^q},$$

it follows that

$$\begin{aligned}
\eta_n &\leq \phi + \frac{1}{\phi^q}\left(\phi + \frac{\eta^q_{n-2(m+1)}}{\omega^q_{n-m-2}}\right) \\
&\leq \phi + \frac{1}{\phi^q}\left(\phi + \frac{1}{\phi^q}\eta^q_{n-2(m+1)}\right) \\
&= \phi + \frac{\phi}{\phi^q} + \frac{1}{\phi^{2q}}\eta^q_{n-2(m+1)} \\
&= \phi + \frac{\phi}{\phi^q} + \frac{1}{\phi^{2q}}\left(\phi + \frac{\eta^q_{n-3(m+1)}}{\omega^q_{n-2m-3}}\right) \\
&\leq \phi + \frac{\phi}{\phi^q} + \frac{\phi}{\phi^{2q}} + \frac{1}{\phi^{3q}}\eta^q_{n-3(m+1)} \\
&\vdots \\
&\leq \phi + \frac{\phi}{\phi^q} + \frac{\phi}{\phi^{2q}} + \cdots + \frac{\phi}{\phi^{(k-1)q}} + \frac{1}{\phi^{kq}}\eta^q_{n-k(m+1)} \\
&= \phi\left[1 \times \frac{1-(\frac{1}{\phi^q})^k}{1-\frac{1}{\phi^q}}\right] + \frac{1}{\phi^{kq}}\eta^q_{n-k(m+1)} \\
&\leq \frac{\phi^{q+1}}{\phi^q - 1}\left(1 - \left(\frac{1}{\phi^{kq}}\right)\right) + \eta^q_{n-k(m+1)}.
\end{aligned}$$

Given that $\frac{n}{m+1} \leq k \leq \frac{n+m}{m+1}$, we deduce that $-m \leq n - k(m+1) \leq 0$, hence

$$\phi \leq \eta_n \leq \frac{\phi^{q+1}}{\phi^q - 1}\left(1 - \left(\frac{1}{\phi^{kq}}\right)\right) + \frac{1}{\phi^{kq}}\eta^q_i,$$

where $i \in \{-m, -m+1, \cdots, -1, 0\}$. Similarly, we have:

$$\phi \leq \omega_n \leq \frac{\phi^{q+1}}{\phi^q - 1}\left(1 - \left(\frac{1}{\phi^{kq}}\right)\right) + \omega^q_i,$$

$$\phi \leq \xi_n \leq \frac{\phi^{q+1}}{\phi^q - 1}\left(1 - \left(\frac{1}{\phi^{kq}}\right)\right) + \xi^q_i.$$

Let us define

$$M = \frac{\phi^{q+1}}{\phi^q - 1}\left(1 - \left(\frac{1}{\phi^{kq}}\right)\right) + \max\{\eta^q_i, \omega^q_i, \xi^q_i\}, \tag{4.4}$$

where $i \in \{-m, -m+1, \cdots, -1, 0\}$. Consequently, from (4.3) and (4.4), we obtain:

$$\phi \leq \eta_n, \omega_n, \xi_n \leq M.$$

■

4.6 Numerical examples

To validate and illustrate the theoretical results obtained in Sections 2–5, we present a series of numerical examples. These simulations aim to (i) confirm the local stability criterion and the asymptotic behavior of the positive equilibrium, (ii) exhibit oscillatory regimes when the hypotheses of Theorem 4.4.1 are satisfied, and (iii) demonstrate boundedness and persistence as stated in Theorem 4.5.1. For several choices of the parameters m, q and ϕ, and for representative initial data, we iterate system (4.1) numerically and display the time-series and phase-portrait plots. Each example below specifies the parameter values and initial conditions, and we comment on how the observed dynamics compare with the analytical predictions (convergence to the equilibrium $\phi + 1$, oscillatory behaviour, or persistence within a bounded band). Figures referenced in the examples illustrate typical trajectories and phase portraits used to support the analytical statements.

Example 4.6.1 *Consider system (4.1) with parameters $m = 3$, $q = 1.2$, and $\phi = 1.6$. So*

$$\begin{cases} \eta_{n+1} = \phi + \dfrac{\eta_{n-3}^{1.2}}{\omega_n^{1.2}}, \\ \omega_{n+1} = \phi + \dfrac{\omega_{n-3}^{1.2}}{\xi_n^{1.2}}, \\ \xi_{n+1} = \phi + \dfrac{\xi_{n-3}^{1.2}}{\eta_n^{1.2}}. \end{cases} \tag{4.5}$$

In this case, the condition $\phi > 2q - 1 = 0.4$ is satisfied, and therefore the equilibrium point

$$(\bar{\eta}, \bar{\omega}, \bar{\xi}) = (\phi + 1, \phi + 1, \phi + 1) = (2.6, 2.6, 2.6)$$

is locally asymptotically stable according to Theorem 4.2.1.

For the initial conditions

$$\begin{aligned} \eta_{-3} &= 0.6, & \omega_{-3} &= 1.2, & \xi_{-3} &= 0.9, \\ \eta_{-2} &= 2.6, & \omega_{-2} &= 2.7, & \xi_{-2} &= 2.9, \\ \eta_{-1} &= 1.7, & \omega_{-1} &= 0.8, & \xi_{-1} &= 3.5, \\ \eta_{0} &= 3.2, & \omega_{0} &= 2.2, & \xi_{0} &= 1.0. \end{aligned}$$

The numerical simulation confirms that all trajectories converge to the positive equilibrium (2.6,2.6,2.6), as illustrated in Figure 4.1. This asymptotic behavior is further emphasized by the three-dimensional phase portrait in Figure 4.2, which clearly depicts the trajectories spiraling towards the common equilibrium point in the state space.

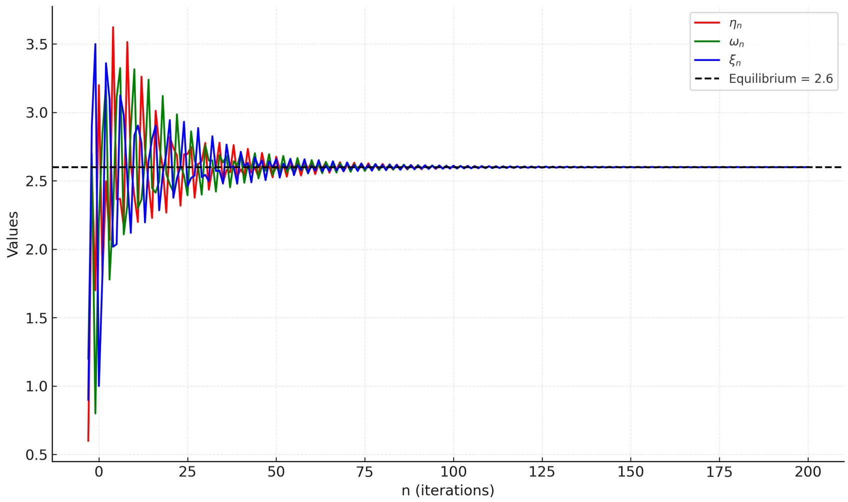


Figure 4.1: The stability of positive equilibrium $(2.6, 2.6, 2.6)$ of system (4.5)

Example 4.6.2 *Consider system (4.1) with parameters $m = 2$, $q = 1.5$, and $\phi = 1$. So*

$$\begin{cases} \eta_{n+1} = \phi + \dfrac{\eta_{n-2}^{1.5}}{\omega_n^{1.5}}, \\ \omega_{n+1} = \phi + \dfrac{\omega_{n-2}^{1.5}}{\xi_n^{1.5}}, \\ \xi_{n+1} = \phi + \dfrac{\xi_{n-2}^{1.5}}{\eta_n^{1.5}}. \end{cases} \tag{4.6}$$

In this case, the condition $0 < \phi < 2q - 1 = 2$ holds. Hence, by the second part of Theorem 4.2.1,

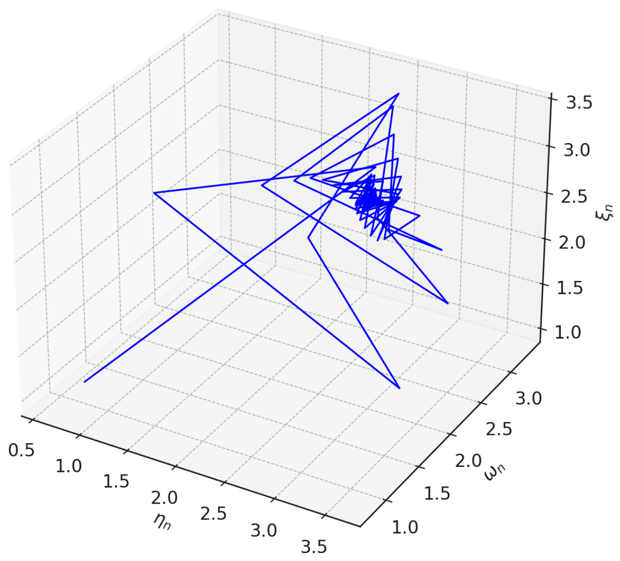


Figure 4.2: 3D phase portrait of the Numerical Solution of the System (4.5)

when $0 < \phi < 2q - 1$, *the positive equilibrium* $(\phi + 1, \phi + 1, \phi + 1)$ *is unstable. For the initial conditions*

$$\begin{aligned} \eta_{-2} &= 0.45, & \omega_{-2} &= 0.90, & \xi_{-2} &= 0.70, \\ \eta_{-1} &= 1.20, & \omega_{-1} &= 1.05, & \xi_{-1} &= 1.70, \\ \eta_{0} &= 2.00, & \omega_{0} &= 2.70, & \xi_{0} &= 1.90. \end{aligned}$$

The following figure confirms that the solution of the system is unstable.

Example 4.6.3 *Consider system* (4.1) *with parameters* $m = 3$, $q = 1$, *and* $\phi = 1$. *So*

$$\begin{cases} \eta_{n+1} = \phi + \dfrac{\eta_{n-3}}{\omega_n}, \\ \omega_{n+1} = \phi + \dfrac{\omega_{n-3}}{\xi_n}, \\ \xi_{n+1} = \phi + \dfrac{\xi_{n-3}}{\eta_n}. \end{cases} \tag{4.7}$$

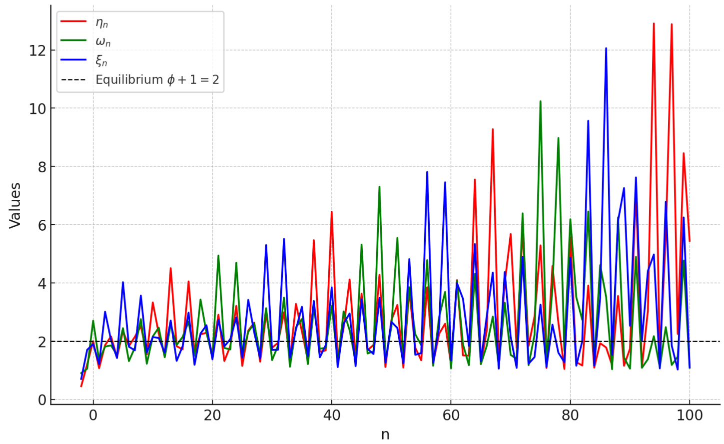


Figure 4.3: The instability of positive equilibrium $(2, 2, 2)$ for system (4.6)

Let the initial values be chosen as follows:

$$\begin{aligned}
\eta_{-3} &= 1.5, & \omega_{-3} &= 1.3, & \xi_{-3} &= 1.4,\\
\eta_{-2} &= 2.3, & \omega_{-2} &= 2.4, & \xi_{-2} &= 2.2,\\
\eta_{-1} &= 1.2, & \omega_{-1} &= 1.1, & \xi_{-1} &= 1.3,\\
\eta_{0} &= 1.4, & \omega_{0} &= 1.7, & \xi_{0} &= 1.5.
\end{aligned}$$

At $n_0 = 0$, we have:

$$\eta_0, \eta_2 < 2 \leq \eta_1, \eta_3, \quad \omega_0, \omega_2 < 2 \leq \omega_1, \omega_3, \quad \xi_0, \xi_2 < 2 \leq \xi_1, \xi_3.$$

Thus, condition (i) of Theorem 4.4.1 is satisfied. Consequently, all components $\{\eta_n\}$, $\{\omega_n\}$, $\{\xi_n\}$ of the solution are oscillatory about the equilibrium point $\phi + 1 = 2$.

Example 4.6.4 *Consider system (4.1) with parameters $m = 3$, $q = 1$, and $\phi = 1.10$. so*

$$\begin{cases}
\eta_{n+1} = \phi + \dfrac{\eta_{n-3}}{\omega_n},\\
\omega_{n+1} = \phi + \dfrac{\omega_{n-3}}{\xi_n},\\
\xi_{n+1} = \phi + \dfrac{\xi_{n-3}}{\eta_n}.
\end{cases} \tag{4.8}$$

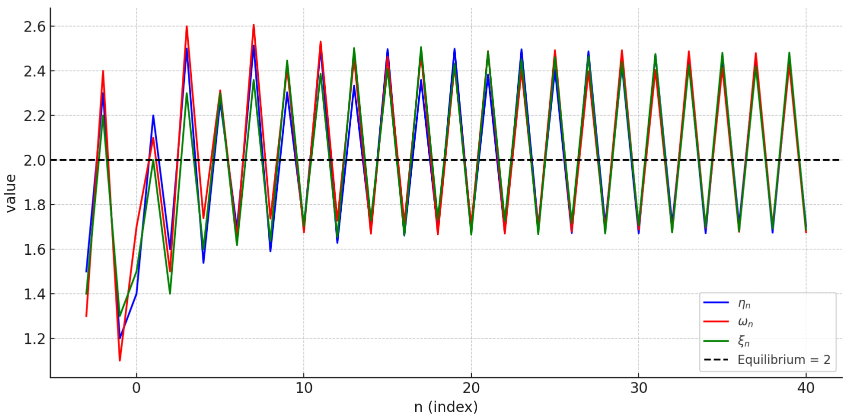


Figure 4.4: The oscillatory solution of system (4.7)

We have

$$\phi^q = 1.10 > 1,$$

and therefore the hypothesis of the Theorem 4.5.1 is satisfied. Choose positive initial data for indices $-3, -2, -1, 0$, *for instance*

$$\begin{aligned} \eta_{-3} &= 0.80, & \omega_{-3} &= 1.00, & \xi_{-3} &= 1.05, \\ \eta_{-2} &= 1.20, & \omega_{-2} &= 0.70, & \xi_{-2} &= 0.85, \\ \eta_{-1} &= 0.90, & \omega_{-1} &= 1.30, & \xi_{-1} &= 1.15, \\ \eta_{0} &= 1.10, & \omega_{0} &= 0.95, & \xi_{0} &= 0.90. \end{aligned}$$

By the Theorem 4.5.1, every positive solution with these parameters remains bounded and persists. To illustrate this numerically, we iterated the system 200 steps forward from the above initial data. The computed ranges of the iterates were (over all computed indices)

$$0.80 \le \eta_n \le 2.33,$$

$$0.70 \le \omega_n \le 2.46,$$

$$0.85 \le \xi_n \le 2.36,$$

and the sequences exhibit damped oscillations approaching a common oscillatory band around approximately 2.10. *These numerical observations are consistent with boundedness and persis-*

tence stated in the theorem.

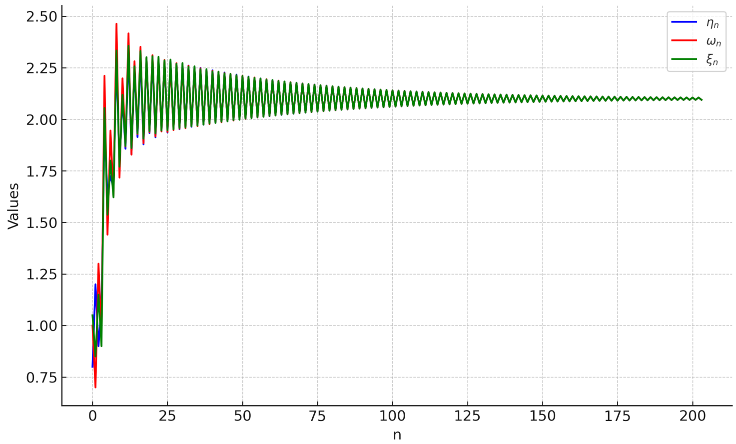


Figure 4.5: Persistence of the solution of system (4.8)

4.7 Conclusion

Chapter 4 advances the theoretical development of the dissertation by transitioning from scalar fuzzy difference equations to multidimensional nonlinear fuzzy dynamical systems. The chapter focuses on the qualitative analysis of a three–dimensional nonlinear rational fuzzy system of the form

$$\begin{cases} x_{n+1} = F(x_n, y_n, z_n), \\ y_{n+1} = G(x_n, y_n, z_n), \qquad n = 0, 1, \ldots, \\ z_{n+1} = H(x_n, y_n, z_n), \end{cases}$$

where the components F, G, and H are rational expressions involving positive fuzzy parameters and positive fuzzy initial conditions. This transition to a higher–dimensional framework allows the chapter to capture richer dynamical behaviors and to explore how uncertainty propagates across interacting fuzzy state variables.

The chapter begins by extending the θ-cut methodology used in earlier chapters to multidimensional settings. Each fuzzy vector is represented as a pair of crisp interval vectors, and the three–dimensional fuzzy system is translated into a coupled system of interval recurrences. The chapter develops the necessary theoretical tools to guarantee existence and uniqueness of fuzzy solutions in this multidimensional environment. Special emphasis is placed on the monotonicity properties of the component functions F, G, and H, as these properties allow the interval endpoints at each θ level to evolve in an ordered manner, ensuring the well-posedness of the fuzzy system.

A substantial portion of the chapter is devoted to the equilibrium analysis. The system of algebraic equations

$$\begin{cases} \bar{x} = F(\bar{x}, \bar{y}, \bar{z}), \\ \bar{y} = G(\bar{x}, \bar{y}, \bar{z}), \\ \bar{z} = H(\bar{x}, \bar{y}, \bar{z}), \end{cases}$$

is shown to have a unique positive equilibrium under natural assumptions on the parameters. The chapter then applies the theory of the Jacobian matrix to study local stability. The equilibrium is linearized at each θ level, yielding interval approximations of the partial derivatives associated with the system. By analyzing the spectral radius of the corresponding Jacobian matrices, the chapter derives explicit criteria ensuring that the fuzzy system is locally asymptotically stable. These results generalize the classical linearization principle to the fuzzy multidimensional setting.

The chapter also investigates global qualitative properties of the system. By constructing suitable invariant fuzzy regions and employing coupled monotonicity conditions, it establishes sufficient conditions under which every positive fuzzy trajectory enters and remains in a compact invariant set. Within this region, the chapter proves that the trajectories converge to the unique fuzzy equilibrium. This global convergence result demonstrates that the interaction of nonlinear rational terms with fuzziness does not destabilize the system, but instead leads to dynamics that remain tightly controlled within predictable bounds.

A further contribution is the analysis of the rate of convergence of the trajectories toward the equilibrium. Building upon the bounds imposed by the Jacobian matrices, the chapter identifies conditions guaranteeing geometric convergence of the interval endpoints at each θ level. The results reveal that the multidimensional structure introduces additional coupling effects that influence the speed of convergence, and that the fuzziness inherent in the initial conditions does not prevent the system from converging rapidly once the trajectories enter the invariant region.

Finally, the chapter presents several numerical simulations designed to confirm and illustrate the theoretical findings. Simulations highlight how the fuzzy state variables interact dynamically, how the θ-cut intervals contract over time, and how multidimensional effects shape the flow of uncertainty through the system. These examples visually demonstrate that the theoretical criteria for boundedness, stability, and convergence are sharp and accurately predict the behavior of the system under various parameter configurations.

In summary, Chapter 4 forms the culmination of the dissertation's theoretical framework. It generalizes the scalar fuzzy models of Chapters 2 and 3 to a fully interacting three–dimensional fuzzy system, thereby showcasing the robustness of the analytical tools developed throughout the dissertation. The chapter also lays the foundation for future research directions, including the study of higher-dimensional fuzzy systems, systems with delays, and fuzzy control applications. As such, it serves both as an endpoint of the qualitative analysis presented in this work and as a gateway to broader investigations in fuzzy dynamical systems.

Chapter 5

General conclusion and perspectives

This thesis presented a detailed investigation of the dynamics and qualitative behavior of certain Fuz. Diff. Eqs., together with a higher-order system of nonlinear difference equations. By means of a rigorous analysis, we highlighted the essential properties of their solutions.

In the second chapter, we applied the concept of g-division for Fuz. Nums. to study the qualitative behavior of higher-order Fuz. Diff. Eqs. of the form

$$y_{n+1} = \beta + \frac{\gamma y_n}{y_{n-\kappa}^2}, \quad n = 0, 1, \dots,$$

where β, γ, and $y_{-\kappa}, y_{-\kappa+1}, \dots, y_0$, with $\kappa \in \{2, 3, \dots\}$, were assumed to be Pos. Fuz. Nums..

For the Fuz. Diff. Eq. (2.1), we showed that there exists a unique Pos. Fuz. Sol. y_n under positive initial conditions $y_{-\kappa}, y_{-\kappa+1}, \dots, y_0$. Moreover, we established the boundedness and persistence of this solution in different cases. We also demonstrated that y_n converges to a unique equilibrium point y, which exists under suitable assumptions, as $n \to \infty$. In addition, we investigated the rate of convergence of the Fuz. Sol.. Finally, we provided two numerical examples in order to illustrate the theoretical results.

In the third chapter, we dealt with the dynamics of nonlinear higher-order Fuz. Diff. Eqs. of the form

$$\omega_{n+1} = \frac{A\omega_{n-k}}{1+\omega_{n-1}^{p}}, \quad n \in \mathbb{N}_0,\ k \in \mathbb{N}_2,$$

where A and $\omega_{-k}, \omega_{-k+1}, \ldots, \omega_0$ were taken as Pos. Fuz. Nums..

Through rigorous analysis, we established the existence, positivity, and uniqueness of the solutions. Furthermore, we obtained sufficient conditions for qualitative properties such as boundedness, persistence, and convergence of Pos. Fuz. Sols.. Several simulation examples supported the theoretical findings and showed their applicability. Thus, our study enriched the theoretical framework of Fuz. Diff. Eqs. and provided insights into their dynamical features.

In the fourth chapter, we investigated a three-dimensional nonlinear system of difference equations with arbitrary constants and non-identical parameters,

$$\begin{cases} \eta_{n+1} = \phi + \dfrac{\eta_{n-m}^{q}}{\omega_n^{q}}, \\ \omega_{n+1} = \phi + \dfrac{\omega_{n-m}^{q}}{\xi_n^{q}}, \\ \xi_{n+1} = \phi + \dfrac{\xi_{n-m}^{q}}{\eta_n^{q}}, \end{cases}$$

where $m \geq 2$, $\eta_i, \omega_i, \xi_i \in]0, +\infty[$ for $i \in \{-m, -m+1, \ldots, -1, 0\}$, $\alpha > 0$, and $q \geq 1$.

We analyzed the local dynamics of the system by linearization around its equilibrium point. Several sufficient conditions were formulated to describe its asymptotic behavior.

The results presented in this dissertation open several promising avenues for future research, both at the theoretical and applied levels. Although the qualitative analysis of nonlinear fuzzy difference equations has progressed significantly, many aspects of this field remain largely unexplored and offer rich potential for further investigation. The following directions outline some of the most relevant and meaningful extensions that may be pursued in future work.

First, the analytical framework developed in this dissertation may be extended to higher-order and more structurally complex fuzzy difference equations, particularly those involving transcendental terms such as exponential or logarithmic components, or equations with time-varying fuzzy parameters. Such generalizations would enhance the applicability of fuzzy models to real-world phenomena where uncertainty evolves dynamically over time, including economic processes, biological growth models, and engineering systems subject to fluctuating conditions.

Second, an important direction for further study lies in the investigation of multidimensional fuzzy systems with strong nonlinear interactions. While the present work focuses on scalar and three-dimensional structures, many real systems such as ecological networks, multi-agent control systems, or biochemical pathways exhibit interactions among several uncertain variables. Developing analytical tools capable of handling highly coupled fuzzy systems, and generalizing existing stability and convergence criteria to higher dimensions, would significantly broaden the theoretical foundation of fuzzy dynamical systems.

Third, the analysis of global stability for nonlinear fuzzy systems remains a central challenge. Much of the existing theory relies on local linearization or the construction of invariant fuzzy intervals. Future work may focus on developing new frameworks based on fuzzy Lyapunov functions, geometric methods, or topological approaches to better understand the long-term behavior of solutions without depending solely on local approximations. Such global techniques could reveal deeper insights into the mechanisms governing stability and bifurcations in fuzzy environments.

Another promising avenue involves the study of fuzzy difference and differential equations with variable delays, which naturally arise in systems with memory or feedback, such as neural processes, delayed population dynamics, and digital control systems. Delay dependent fuzzy models introduce a dual form of complexity uncertainty in both state and time and may exhibit new dynamical behaviors that merit detailed investigation.

In summary, while this dissertation provides a substantial contribution to the study of nonlinear fuzzy difference equations, it also highlights the vast potential for further research. The directions outlined above represent only a portion of the numerous possibilities available for extending and applying the theoretical foundations established herein.

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